Tutorial 1

Problem 1.1

Prove that: $\frac{a^2+b^2}{2} \ge ab$

Solution:

$\frac{a^2+b^2}{2} \ge ab \Leftrightarrow a^2+b^2 \ge 2ab \Leftrightarrow a^2-2ab+b^2 \ge 0 \Leftrightarrow (a-b)^2 \ge 0$ - true!

Problem 1.2

Prove that: $\left|x^2 - xy + y^2\right| \ge |xy|$

Solution:

Step 1: $x^2-xy+y^2 = x^2-xy+\left(\frac{1}{2}y\right)^2+\frac{3}{4}y^2 = \left(x-\frac{1}{2}y\right)^2+\frac{3}{4}y^2 \ge 0$

Step 2: $\left|x^2 - xy + y^2\right| \ge |xy| \Leftrightarrow x^2-xy+y^2 \ge |xy|$

Step 3:

If $xy>0$ then $x^2-xy+y^2 \ge |xy| \Leftrightarrow x^2-xy+y^2 \ge xy \Leftrightarrow x^2-2xy+y^2 \ge 0 \Leftrightarrow (x-y)^2 \ge 0$ - true!

If $xy<0$ then $x^2-xy+y^2 \ge |xy| \Leftrightarrow x^2-xy+y^2 \ge -xy \Leftrightarrow x^2+y^2 \ge 0$ - true!

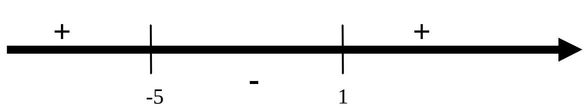
Problem 1.3

Solve: $(x-1)(x+5)<0$

Solution:

A bad (but correct) solution: $\begin{cases} x-1<0 \\ x+5>0 \end{cases} or \begin{cases} x-1>0 \\ x+5<0 \end{cases} \Rightarrow x \in (-5;1)$

A good solution:



$\Rightarrow x \in (-5;1)$

The rules:

1. Begin from "plus" on the right side.
2. Change a sign for every odd degree, and don't change for every even degree.

Problem 1.4

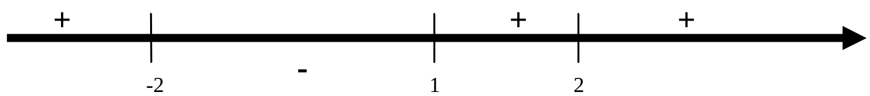
Solve: $x^4+5x^3-2x^2+8x-8<8x^3-4x$

Solution:

$x^4+5x^3-2x^2+8x-8<8x^3-4x \iff x^4-3x^3-2x^2+12x-8<0$

Need to check divisors of 8. 1, 2, and -2 solve the polynomial, so $(x-1)$, $(x-2)$ and $(x+2)$ are the factors.

Using a polynomial division we obtain: $x^4-3x^3-2x^2+12x-8=(x-1)(x-2)^2(x+2)$.



So, $x\in(-2;1)$

Problem 1.5

Solve: $\dfrac{x+3}{x-1}>2$

Solution:

Wrong way: $\dfrac{x+3}{x-1}>2 \iff x+3>2(x-1) \iff x<5$

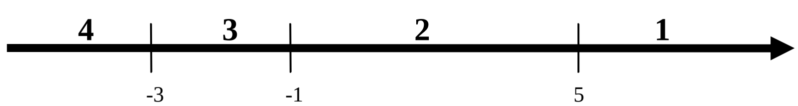
Right way:

$\dfrac{x+3}{x-1}>2 \iff \dfrac{x+3}{x-1}-2>0 \iff \dfrac{x+3-2(x-1)}{x-1}>0 \iff \dfrac{-x+5}{x-1}>0 \iff \dfrac{x-5}{x-1}<0 \iff x\in(1;5)$

Problem 1.6

Solve: $|x-5|+|x+3|+|x+1|\le 15$

Solution:



$[5;+\infty)$: $x-5+x+3+x+1\le 15 \iff 3x\le 16$. So, $x\in\left[5;\dfrac{16}{3}\right]$

$[-1;+5)$: $-x+5+x+3+x+1\le 15 \iff x\le 6$. So, $x\in[-1;+5)$

$[-3;-1)$: $-x+5+x+3-x-1\le 15 \iff x\ge -8$. So, $x\in[-3;-1)$.

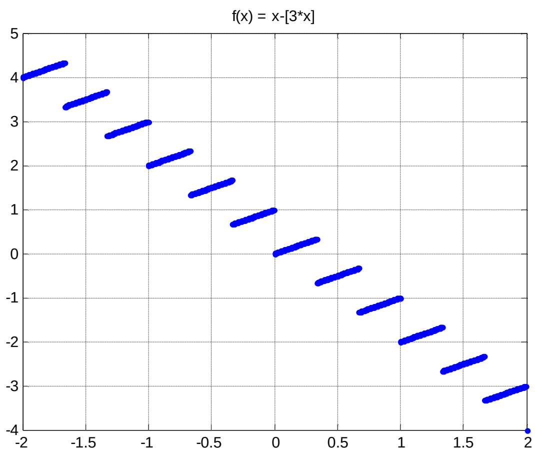
$(-\infty;-3)$: $-x+5-x-3-x-1\le 15 \iff x\ge -\dfrac{14}{3}$. So, $x\in\left[-\dfrac{14}{3};-3\right)$

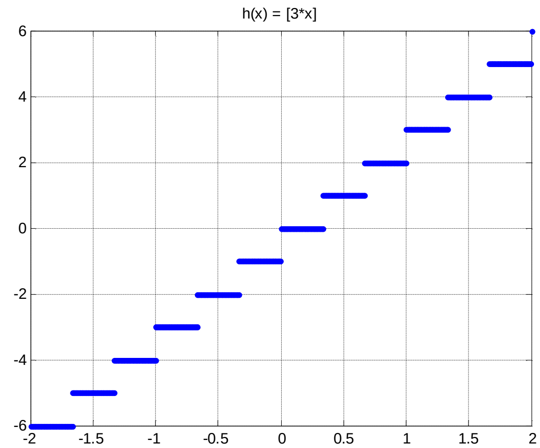
The answer is: $x\in\left[-\dfrac{14}{3};\dfrac{16}{3}\right]$.

Problem 1.7

Sketch the graph of $f(x) = x - [3x]$, $h(x) = [3x]$.

Solution:

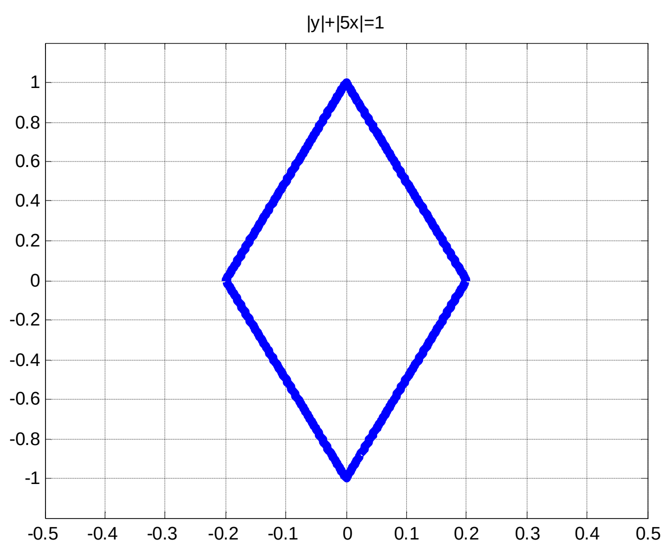




Problem 1.8

Sketch the graph of $|y| + |5x| = 1$

Solution:



Problem 1.9

Determine the domain of the function: $\dfrac{1}{\ln(x^2+3x-4)}$

Solution:

$$\begin{cases} x^2+3x-4>0 \\ x^2+3x-4\neq 1 \end{cases} \Leftrightarrow$$

$$\begin{cases} (x-1)(x+4)>0 \\ x^2+3x-5\neq 0 \end{cases} \Leftrightarrow \begin{cases} x\in(-\infty;-4)\cup(1;+\infty) \\ x\neq \dfrac{-3\pm\sqrt{29}}{2}\approx 1.19;-4.19 \end{cases} \Leftrightarrow$$

$$x\in\left(-\infty;\frac{-3-\sqrt{29}}{2}\right)\cup\left(\frac{-3-\sqrt{29}}{2};-4\right)\cup\left(1;\frac{-3+\sqrt{29}}{2}\right)\cup\left(\frac{-3+\sqrt{29}}{2};+\infty\right)$$

Problem 1.10

Sketch the graph of the function: $y=-x^2+5x-3$

Solution:

$$y=-x^2+5x-3=-\left(x^2-5x+3\right)=-\left(x^2-5x+6.25\right)+3.25=-\left(x-2.5\right)^2+3.25$$