Tutorial 10

Problem 10.1

Prove using the Riemann sums that $\int_a^b x^2 dx = \frac{1}{3}\left(b^3 - a^3\right)$, $a, b > 0$

Solution:

- x^2 is an increasing function on the interval $[a;b]$, that is why the minimum point for any interval is attained on the left side and the maximum point is attained on the right side of the interval.
- Choose the partition $P = \{x_0, x_1, ..., x_n\}$, when $x_0 = a,\ x_n = b$.
- $U_f(P) = x_1^2(x_1 - x_0) + x_2^2(x_2 - x_1) + ... + x_n^2(x_n - x_{n-1})$
- $L_f(P) = x_0^2(x_1 - x_0) + x_1^2(x_2 - x_1) + ... + x_{n-1}^2(x_n - x_{n-1})$
- For each index i , $1 \le i \le n$, $x_{i-1}^2 \le \frac{1}{3}\left(x_{i-1}^2 + x_{i-1}x_i + x_i^2\right) \le x_i^2$
- So, $(x_i - x_{i-1})x_{i-1}^2 \le \frac{1}{3}(x_i - x_{i-1})\left(x_{i-1}^2 + x_{i-1}x_i + x_i^2\right) \le (x_i - x_{i-1})x_i^2 \quad \Leftrightarrow$

  $(x_i - x_{i-1})x_{i-1}^2 \le \frac{1}{3}\left(x_i^3 - x_{i-1}^3\right) \le (x_i - x_{i-1})x_i^2$
- $\sum_{i=1}^{n} \frac{1}{3}\left(x_i^3 - x_{i-1}^3\right) = \frac{1}{3}\left(x_n^3 - x_0^3\right) = \frac{1}{3}\left(a^3 - b^3\right)$
- $L_f(P) \le \frac{1}{3}\left(a^3 - b^3\right) \le U_f(P)$. That is why $\int_a^b x^2 dx = \frac{1}{3}\left(b^3 - a^3\right)$

Problem 10.2

Which continuous functions f defined on $[a;b]$ have the property that all lower sums $L_f(P)$ are equal?

Solution:

- Assume the function is not constant on some interval $[c;d] \subseteq [a;b]$.
- Denote $M = \min_{x \in [c;d]} f(x)$.
- Then exists an interval $[c_1;d_1] \subseteq [c;d]$ such that for any $x \in [c_1;d_1]$ exists $f(x) > M$ (why?)
- Then for partition $[a;c_1], [c_1;d_1], [d_1;b]$ the lower sum is less than for the partition $[a;b]$.
- Contradiction. That is why the function is constant.

Problem 10.3

Let f be a function such that f' is continuous on $[a;b]$. Show that $\int_a^b f(t)f'(t)dt = \frac{1}{2}[f'(b) - f'(a)]$

Solution:

- $\frac{1}{2}f^2(t) = \frac{1}{2}2f(t)f'(t) = f(t)f'(t)$
- That is why $\int_a^b f(t)f'(t)dt = \frac{1}{2}f^2(t)\Big|_a^b = \frac{1}{2}[f'(b) - f'(a)]$

Problem 10.4

Given a continuous function, set $F(x) = \int_0^x t \cdot f(t)dt$. Find $F'(x)$.

Solution:

Assume $g(t) = \int t \cdot f(t)dt$.

$$F'(x) = \frac{d}{dx}\int_0^x t \cdot f(t)dt = \frac{d}{dx}([g(x) - g(0)]) = \frac{d}{dx}(g(x)) - 0 = x \cdot f(x)$$

Problem 10.5

Calculate $\int [f(x)g''(x) - g(x)f''(x)]dx$

Solution:

$$\int [f(x)g''(x) - g(x)f''(x)]dx = \int [f(x)g''(x) + f'(x)g'(x) - f'(x)g'(x) - g(x)f''(x)]dx =$$
$$= f(x)g'(x) - f'(x)g(x)$$

Problem 10.6

Calculate $\int \sin x \cos^2 x\, dx$

Solution:

$$\int \sin x \cos^2 x\, dx = -\int \cos^2 x (\cos x)' dx = -\frac{\cos^3 x}{3}$$

or another way:

$$\int \sin x \cos^2 x\, dx = \int \sin x(1 - \sin^2 x)dx = \int \sin x\, dx - \int \sin^3 x\, dx = -\cos x - \frac{1}{4}\int 3\sin x - \sin 3x\, dx =$$
$$= -\cos x + \frac{3}{4}\cos x - \frac{1}{12}\cos 3x = -\frac{1}{4}\cos x - \frac{1}{12}\cos 3x$$

Problem 10.7

Calculate $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}dx$

Solution:

$$\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}dx = -\int \sin\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)' dx = \cos\left(\frac{1}{x}\right)$$

Problem 10.8

Let f be a continuous function, c a real number. Show that

a. $\int_{a+c}^{b+c} f(x-c)dx = \int_a^b f(x)dx$

b. $\frac{1}{c}\int_{ac}^{bc} f\left(\frac{x}{c}\right)dx = \int_a^b f(x)dx$

Solution:

a.

- $\int_{a+c}^{b+c} f(x-c)dx = \int_{a+c}^{b+c} f(x-c)(x-c)'dx = \int_a^b f(y)dy$, when $y = x-c$

b.

- $\frac{1}{c}\int_{ac}^{bc} f\left(\frac{x}{c}\right)dx = \frac{1}{c}c\int_{ac}^{bc} f\left(\frac{x}{c}\right)\left(\frac{x}{c}\right)' dx = \int_a^b f(y)dy$, when $y = \frac{x}{c}$.

Problem 10.9

Evaluate: $\int_0^{\pi} x\cos x^2 dx$

Solution:

- $\int_0^{\pi} x\cos x^2 dx = \frac{1}{2}\int_0^{\pi}\left(x^2\right)' \cos x^2 dx = \frac{1}{2}\cos x^2\Big|_0^{\pi} = \frac{1}{2}\cos\pi^2 - \frac{1}{2}$

Problem 10.10

Calculate: $\int 2x\sqrt{x-1}\,dx$

Solution:

- Substitute $y = x-1$

- $\int 2x\sqrt{x-1}\,dx = \int 2(y+1)\sqrt{y}\,dy = \int 2y^{\frac{3}{2}}\,dy + \int 2\sqrt{y}\,dy = \frac{4}{5}y^{\frac{5}{2}} + \frac{4}{3}y^{\frac{3}{2}} = \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}}$

Problem 10.11

Differentiate $\int_0^{\sin x} \frac{t^2+\cos(t^7)}{1+t^4}\,dt$

Solution:

$$\int_0^{\sin x} \frac{t^2+\cos(t^7)}{1+t^8}\,dt = \int_0^{\sin x} \frac{t^2+\cos(t^7)}{1+t^8}\,dt = \frac{\sin^2 x+\cos(\sin^7 x)}{1+\sin^8 x}\cos x$$