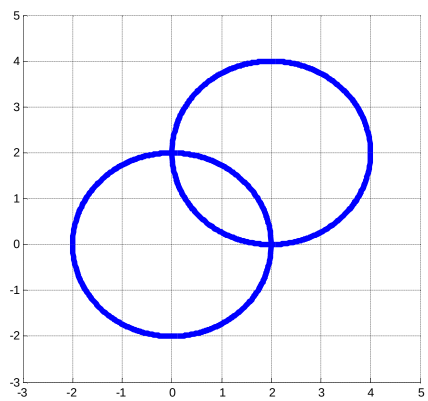
Tutorial 11

Problem 11.1

Represent the area determined by the intersection of the circles $x^2+y^2=4$ and $(x-2)^2+(y-2)^2=4$ by one or more integrals.

Solution:

- The lower part of the circle $(x-2)^2+(y-2)^2=4$ is $y=2-\sqrt{4-(x-2)^2}$.
- The upper part of the circle $x^2+y^2=4$ is $y=\sqrt{4-x^2}$.
- $\int_0^2 \sqrt{4-x^2}-\left(2-\sqrt{4-(x-2)^2}\right)dx$



Problem 11.2

The base of a solid is the disk bounded by the circle $x^2+y^2=r^2$. Find the volume of the solid given that cross sections perpendicular to the x-axis are: (a) squares; (b) equilateral triangles.

Solution:

a.

- It means x goes from $-r$ to r.
- For every value of x square area is $\left(2\sqrt{r^2-x^2}\right)^2$
- $V=\int_{-r}^{r}\left(2\sqrt{r^2-x^2}\right)^2 dx=8\int_0^r\left(r^2-x^2\right)^2 dx=8\left[r^2x-\frac{1}{3}x^3\right]_0^r=\frac{16}{3}r^3$

b.

- $V=\int_{-r}^{r}\frac{\sqrt{3}}{4}\left(2\sqrt{r^2-x^2}\right)^2 dx=\frac{4\sqrt{3}}{3}r^3$

Problem 11.3

Use the shell method to find the volume enclosed by the surface obtained by revolving the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the y axis ($a, b > 0$)

Solution:

- $b^2x^2 + a^2y^2 = a^2b^2 \Leftrightarrow \frac{x}{a^2}^2 + \frac{y}{b^2}^2 = 1$
- In the first quarter we get: $y = b\sqrt{1 - \frac{x^2}{a^2}}$
- $\frac{V}{2} = \int_0^a 2\pi x \left[2b\sqrt{1-\frac{x^2}{a^2}}\right] dx = \frac{2\pi b}{a}\int_0^a x\left(a^2 - x^2\right)^{\frac{1}{2}} dx = \frac{2\pi b}{a}\left[-\frac{1}{3}\left(a^2 - x^2\right)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}\pi a^2 b$
- $V = \frac{4}{3}\pi a^2 b$

Problem 11.4

a. Let $f(x) = \frac{1}{3}x^3 + x^2 + kx$, k a constant. For what values of k is f one-to-one?

b. Let $g(x) = x^3 + kx^2 + x$, k a constant. For what values of k is g one-to-one?

Solution:

a. $f'(x) = x^2 + 2x + k$: f will be strongly increasing on $(-\infty;\infty)$ if f' doesn't change sign. This will occur if the discriminant of f', namely $4-4k$ is non positive. $4-4k \le 0 \Rightarrow k \ge 1$

b. $g(x) = 3x^2 + 2kx + 1$: g will be increasing on $(-\infty;\infty)$ if g' doesn't change sign. This will occur if the discriminant of g', namely $4k^2 - 12$ is non-positive. $4k^2 - 12 \le 0 \Rightarrow k^2 \le 3 \Rightarrow -\sqrt{3} \le k \le \sqrt{3}$

Problem 11.5

Set $f(x) = \int_2^x \sqrt{1+t^2}\,dt$

a. Show that f has an inverse.

b. Find $\left(f^{-1}\right)'(0)$

Solution:

a. $f'(x) = \left(\int_2^x \sqrt{1+t^2}\,dt\right)' = \sqrt{1+x^2}$, so f is always increasing, hence one to one.

b. $\left(f^{-1}\right)'(0)=\dfrac{1}{f'\left(f^{-1}(0)\right)}=\dfrac{1}{f'(2)}=\dfrac{1}{\sqrt{1+2^2}}=\dfrac{1}{5}$

Problem 11.6

Solve the equation for x: $(2-\ln x)\ln x=0$

Solution:

- $2-\ln x=0$ or $\ln x=0 \Rightarrow x=e^2$ or $x=1$

Problem 11.7

Prove that $\dfrac{1}{2}+...+\dfrac{1}{n}<\ln n<\dfrac{1}{1}+...+\dfrac{1}{n-1}$

Solution:

- $\ln n=\ln n-\ln 1=\int_1^n \dfrac{1}{x}dx$
- Making a partition when every interval is of length 1.
- From the lower sum we obtain: $\ln n=\int_1^n \dfrac{1}{x}dx>\dfrac{1}{2}+...+\dfrac{1}{n}$
- The upper sum: $\ln n=\int_1^n \dfrac{1}{x}dx<\dfrac{1}{1}+...+\dfrac{1}{n-1}$

Problem 11.7

Calculate $\int \dfrac{\sin x-\cos x}{\sin x+\cos x}dx$

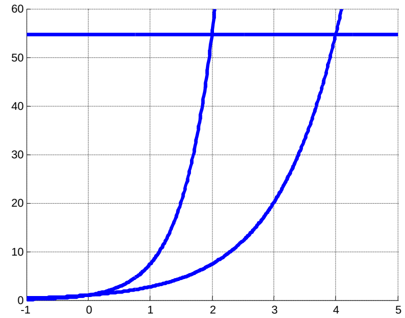
Solution:

- Making a substitution: $u=\sin x+\cos x \Rightarrow du=(\cos x-\sin x)dx$
- $\int \dfrac{\sin x-\cos x}{\sin x+\cos x}dx=\int \dfrac{1}{u}du=\ln u=\ln(\sin x+\cos x)$

Problem 11.8

Find the area of the region bounded by the curves $y = e^{2x}$, $y = e^x$, $y = e^4$.

Solution:



- $A = \int_0^2 \left(e^{2x} - e^x\right) dx + \int_2^4 \left(e^4 - e^x\right) dx = \left(\frac{e^{2x}}{2} - e^x\right)\Big|_0^2 + \left(e^4 x - e^x\right)\Big|_2^4 = \frac{1}{2}\left(3e^4 + 1\right)$