Tutorial 13

Problem 13.1

Find the area of the region, bounded by the curves: $y = e^x$, $y = e^{2x}$, $y = e^4$

Solution:

$$A = \int_0^2 \left(e^{2x} - e^x\right)dx + \int_2^4 \left(e^4 - e^x\right)dx = \left[\frac{1}{2}e^{2x} - e^x\right]_0^2 + \left[e^4 x - e^x\right]_2^4 =$$

$$= \left(\frac{1}{2}e^4 - e^2 - \frac{1}{2} + 1\right) + \left(4e^4 - e^4 - 2e^4 + e^2\right) = \frac{1}{2}\left(3e^4 + 1\right)$$

Problem 13.2

Determine the following:

1. The domain
2. The intervals on which f increases, decreases
3. The extreme values
4. The concavity of the graph and the points of inflection

Then sketch the graph of $f(x) = e^{\frac{1}{x^2}}$

Solution:

- $f(x) = e^{\left(\frac{1}{x}\right)^2}$, $f'(x) = -\dfrac{2}{x^3}e^{\left(\frac{1}{x}\right)^2}$, $f''(x) = \dfrac{6x^2 + 4}{x^6}e^{\left(\frac{1}{x}\right)^2}$
- Domain: $(-\infty; 0) \cup (0; \infty)$
- Increases on $(-\infty; 0)$, decreases on $(0; \infty)$
- No extreme values
- Concave up on $(-\infty; 0)$ and on $(0; \infty)$

Problem 13.3

Calculate $\dfrac{d}{dx}\left[(\ln x)^{\ln x}\right]$

Solution:

- $y = (\ln x)^{\ln x}$
- $\ln y = \ln x\left(\ln(\ln x)\right)$
- $\dfrac{1}{y}\dfrac{dy}{dx} = \ln x\left[\dfrac{1}{x \ln x}\right] + \dfrac{1}{x}\left(\ln(\ln x)\right)$

- $\frac{dy}{dx} = (\ln x)^{\ln x}\left[\ln x\left[\frac{1}{x\ln x}\right] + \frac{1}{x}(\ln(\ln x))\right] = (\ln x)^{\ln x}\left[\frac{1+\ln(\ln x)}{x}\right]$

Problem 13.4

Calculate $\frac{d}{dx}\left[(\sin x)^{\cos x}\right]$

Solution:

- $y = (\sin x)^{\cos x}$
- $\ln y = (\cos x)(\ln(\sin x))$
- $\frac{1}{y}\frac{dy}{dx} = (-\sin x)(\ln(\sin x)) + (\cos x)\left(\frac{1}{\sin x}\right)(\cos x)$
- $\frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - (\sin x)(\ln(\sin x))\right]$

Problem 13.5

Find all the functions f that satisfy the equations for all real t: $f'(t) = t \cdot f(t)$

Solution:

- $f'(t) = t \cdot f(t)$
- $f'(t) - t \cdot f(t) = 0$
- $e^{-\frac{t^2}{2}} f'(t) - te^{-\frac{t^2}{2}} \cdot f(t) = 0$
- $\frac{d}{dt}\left[e^{-\frac{t^2}{2}} f(t)\right] = 0$
- $e^{-\frac{t^2}{2}} f(t) = C \;\Rightarrow\; f(t) = Ce^{\frac{t^2}{2}}$

Problem 13.6

Calculate $\int \frac{\sec^2 x}{9+\tan^2 x}dx$

Solution:

- $\int \frac{\sec^2 x}{9+\tan^2 x}dx$
- Make a substitution $\begin{matrix} u = \tan x \\ du = \sec^2 x\, dx \end{matrix}$

- $\int \frac{\sec^2 x}{9+\tan^2 x}\,dx = \int \frac{1}{9+u^2}\,dx = \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C = \frac{1}{3}\arctan\left(\frac{\tan x}{3}\right) + C$

Problem 13.7

Derive the formula: $\int x^k \ln x\,dx = \frac{x^{k+1}}{k+1}\ln x - \frac{x^{k+1}}{(k+1)^2} + C, k \neq -1$

Solution:

- Make a substitution: $u = \ln x,\ du = \frac{1}{x}dx,\quad v = \frac{1}{k+1}x^{k+1},\ dv = x^k dx$
- $\int x^k \ln x\,dx = \frac{1}{k+1}x^{k+1}\ln x - \int \frac{1}{k+1}x^k\,dx = \frac{1}{k+1}x^{k+1}\ln x - \frac{x^{k+1}}{(k+1)^2} + C$

Problem 13.8

The region between the curve $y = \tan x$ and the x-axis from $x = 0$ to $x = \frac{\pi}{4}$ is revolved about the line $y = -1$.

Find the volume of the resulting solid.

Solution:

- $$V = \int_0^{\frac{\pi}{4}} \pi\left[(\tan x + 1)^2 - 1^2\right]dx = \pi\int_0^{\frac{\pi}{4}}\left[\tan^2 x + 2\tan x\right]dx = \pi\int_0^{\frac{\pi}{4}}\left[\tan^2 x + 2\tan x\right]dx =$$
$$= \pi\int_0^{\frac{\pi}{4}}\left[\sec^2 x + 2\tan x - 1\right]dx = \pi\left[\tan x + 2\ln|\sec x| - x\right]_0^{\frac{\pi}{4}} = \pi\left[\ln 2 + 1 - \frac{\pi}{4}\right]$$