Tutorial 14

Problem 14.1

Calculate: $\int \frac{dx}{e^x\sqrt{e^{2x}-9}}$

Solution:

- $e^x = 3\sec u$
  $e^x dx = 3\sec u \tan u\, du$

- $\int \frac{dx}{e^x\sqrt{e^{2x}-9}} = \int \frac{\tan u du}{3\sec u(3\tan u)} = \frac{1}{9}\int \cos u\, du = \frac{1}{9}\sin u + C = \frac{1}{9}e^{-x}\sqrt{e^{2x}-9} + C$

Problem 14.2

Calculate: $\int \frac{dx}{\left(x^2-4x+4\right)^{\frac{3}{2}}}$

Solution:

- $\frac{1}{\left(x^2-4x+4\right)^{\frac{3}{2}}} = \begin{cases} (x-2)^{-3},\ x>2 \\ (2-x)^{-3},\ x<2 \end{cases}$

- $\int \frac{dx}{\left(x^2-4x+4\right)^{\frac{3}{2}}} = \begin{cases} -\frac{1}{2(x-2)^2}+C,\ x>2 \\ \frac{1}{2(x-2)^2}+C,\ x<2 \end{cases}$

Problem 14.3

Calculate: $\int x\sqrt{6x-x^2-8}\,dx$

Solution:

- $x-3 = \sin u$
  $dx = \cos u\, du$

  $\int x\sqrt{6x-x^2-8}\,dx = \int x\sqrt{1-(x-3)^2}\,dx = \int (3+\sin u)(\cos u)\cos u\, du = \int 3\cos^2 u + \cos^2 u \sin u\, du =$

- $= \int 3\frac{1+\cos 2u}{2} + \cos^2 u \sin u\, du = \frac{3}{2}u + \frac{3}{4}\sin 2u - \frac{1}{3}\cos^3 u + C =$

  $= \frac{3}{2}\arcsin(x-3) + \frac{3}{2}(x-3)\sqrt{6x-x^2-8} - \frac{1}{3}\left(6x-x^2-8\right)^{\frac{3}{2}} + C$

Problem 14.4

Evaluate: $\int_0^2 \frac{x}{x^2+5x+6}dx$

Solution:

- $\int_0^2 \frac{x}{x^2+5x+6}dx = \int_0^2 \frac{x}{(x+2)(x+3)}dx = \int_0^2 \frac{3}{x+2} - \frac{2}{x+3}dx = \left(3\ln|x+3| - 2\ln|x+2|\right)\Big|_0^2 = \ln\frac{125}{108}$

Problem 14.5

Evaluate: $\int_1^3 \frac{x^2-4x+3}{x^3+2x^2+x}dx$

Solution:

- $\int_1^3 \frac{x^2-4x+3}{x^3+2x^2+x}dx = \int_1^3 \frac{x^2-4x+3}{x(x+1)^2}dx = \int_1^3 \frac{3}{x} - \frac{2}{x+1} - \frac{8}{(x+1)^2}dx = \left[3\ln|x| - 2\ln|x+1| + \frac{8}{x+1}\right]_1^3 = \ln\left(\frac{27}{4}\right) - 4$

Problem 14.6

Calculate: $\int \frac{\cos\theta}{\sin^2\theta - 2\sin\theta - 8}d\theta$

Solution:

- $\int \frac{\cos\theta}{\sin^2\theta - 2\sin\theta - 8}d\theta = \frac{1}{6}\int \frac{\cos\theta}{\sin\theta - 4}d\theta - \frac{1}{6}\int \frac{\cos\theta}{\sin\theta + 2}d\theta =$
  $= \frac{1}{6}\ln|\sin\theta - 4| - \frac{1}{6}\ln|\sin\theta + 2| + C = \frac{1}{6}\ln\left|\frac{\sin\theta - 4}{\sin\theta + 2}\right| + C$

Problem 14.7

Calculate: $\int \frac{1}{t\left([\ln t]^2 - 4\right)}dt$

Solution:

- $\int \frac{1}{t\left([\ln t]^2 - 4\right)}dt = \frac{1}{4}\int \frac{1}{t(\ln t - 2)}dt - \frac{1}{4}\int \frac{1}{t(\ln t + 2)}dt = \frac{1}{4}\ln|\ln t - 2| - \frac{1}{4}\ln|\ln t + 2| + C = \frac{1}{4}\ln\left|\frac{\ln t - 2}{\ln t + 2}\right| + C$

Problem 14.8

Evaluate: $\int_0^4 \frac{x^{\frac{3}{2}}}{x+1}dx$

Solution:

- $u^2 = x$
  $2u\,du = dx$
- $\int \frac{x^{\frac{3}{2}}}{x+1}dx = \int \frac{u^3}{u^2+1}2u\,du = \int \frac{2u^4}{u^2+1}du = \int \left[2u^2 - 2 + \frac{2}{u^2+1}\right]du =$
  $= \frac{2}{3}u^3 - 2u + 2arc\tan u + C = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 2arc\tan x^{\frac{1}{2}} + C$
- $\int_0^4 \frac{x^{\frac{3}{2}}}{x+1}dx = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 2arc\tan x^{\frac{1}{2}} + C\Big|_0^4 = \frac{4}{3} + 2\arctan 2$

Problem 14.9

Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos x}dx$

Solution:

- $u = \cos x,\ du = -\sin x dx$
- $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos x}dx = \int_0^{\frac{\pi}{2}} \frac{2\sin x\cos x}{2+\cos x}dx = \int_0^1 \frac{2u}{2+u}du = \int_0^1 \left(2 - \frac{4}{2+u}\right)du = \left[2u - 4\ln|2+u|\right]_0^1 = 2 + 4\ln\left(\frac{2}{3}\right)$

Problem 14.10

Evaluate: $\int_0^{\frac{\pi}{3}} \frac{1}{\sin x - \cos x - 1}dx$

Solution:

- $u = \tan\left(\frac{x}{2}\right),\ dx = \frac{2}{1+u^2}du$
  $\sin x = \frac{2u}{1+u^2},\ \cos x = \frac{1-u^2}{1+u^2}$
- $\int \frac{1}{\sin x - \cos x - 1}dx = \int \frac{1}{\frac{2u}{1+u^2} - \frac{1-u^2}{1+u^2} - 1}\cdot\frac{2}{1+u^2}du = \int \frac{1}{u-1}du = \ln|u-1| + C = \ln\left|\tan\left(\frac{x}{2}\right) - 1\right| + C$

- $\int_0^{\frac{\pi}{3}} \frac{1}{\sin x - \cos x - 1} dx = \ln\left|\tan\left(\frac{x}{2}\right) - 1\right|\Big|_0^{\frac{\pi}{3}} = \ln\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)$

Problem 14.11

Evaluate: $\int_0^1 \arcsin x \, dx$

Solution:

- $\int_0^1 \arcsin x \, dx = x \arcsin x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \frac{\pi}{2} + \sqrt{1-x^2}\Big|_0^1 = \frac{\pi}{2} - 1$

Problem 14.12

Use comparison test to determine whether the integral converges: $\int_1^{\infty} \frac{x}{\sqrt{1+x^5}} dx$

Solution:

- $\int_1^{\infty} \frac{x}{\sqrt{1+x^5}} dx \le \int_1^{\infty} \frac{x}{\sqrt{x^5}} dx = \int_1^{\infty} \frac{1}{\sqrt{x^3}} dx = -\frac{2}{3} x^{-\frac{1}{2}}\Big|_1^{\infty} = \frac{2}{3}$

Problem 14.13

Use comparison test to determine whether the integral converges: $\int_1^{\infty} \frac{\ln x}{x^2} dx$

Solution:

- $\ln x \le \sqrt{x}$ for $x \ge 1$, because $\left(\sqrt{x} - \ln x\right)' = \frac{1}{2\sqrt{x}} - \frac{1}{x} = 0 \Rightarrow x = 4$, so the minimum can be attained

  only at $x = 1$ or $x = 4$. But for those points we obtain that $\sqrt{x} - \ln x > 0$

- $\int_1^{\infty} \frac{\ln x}{x^2} dx \le \int_1^{\infty} \frac{\sqrt{x}}{x^2} dx \le \int_1^{\infty} \frac{1}{\sqrt{x^3}} dx = -\frac{2}{3} x^{-\frac{1}{2}}\Big|_1^{\infty} = \frac{2}{3}$