Tutorial 15

Problem 15.1

Prove that the formula for a Fibonacci element is: $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Solution:

- Firstly, we are going to find two sequences with $a_n = a_{n-1} + a_{n-2}$ but other initial values.
- Assume geometric sequence $1, x, x^2, \ldots$.
- Then $x^{n+2} = x^{n+1} + x^n \Leftrightarrow x^2 = x + 1 \Leftrightarrow x = \frac{1 \pm \sqrt{5}}{2}$
- Now, we will construct the initial values.
- $\begin{cases} a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1 \\ a\left(\frac{1+\sqrt{5}}{2}\right)^2 + b\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1 \end{cases} \Rightarrow \begin{cases} a\left(1+\sqrt{5}\right) + b\left(1-\sqrt{5}\right) = 2 \\ a\left(6+2\sqrt{5}\right) + b\left(6-2\sqrt{5}\right) = 4 \end{cases} \Rightarrow a = -b = \frac{1}{\sqrt{5}}$
- So, the formula is $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Problem 15.2

Prove that $\sqrt{2}$ is irrational.

Solution:

- Assume $\sqrt{2}$ is rational.
- Then $\sqrt{2} = \frac{a}{b}$ for some integer a, b.
- Assume the fraction $\frac{a}{b}$ is reduced. It means $\gcd(a,b) = 1$.
- $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow 2 \mid a \Rightarrow 4 \mid a^2 \Rightarrow 2 \mid b$. Contradiction.

Problem 15.3

Calculate: $\int \frac{1}{2+\sin x}\,dx$.

Solution:

- $u=\tan\left(\frac{x}{2}\right)$, $dx=\frac{2}{1+u^2}du$, $\sin x=\frac{2u}{1+u^2}$

$$\int\frac{1}{2+\sin x}dx=\int\frac{1}{2+\frac{2u}{1+u^2}}\frac{2}{1+u^2}dx=\int\frac{1}{1+u+u^2}du=$$

- $$=\int\frac{1}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}du=\frac{2}{\sqrt{3}}\arctan\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C=\frac{2}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\left(2\tan\left(\frac{x}{2}\right)+1\right)\right)+C$$

Problem 15.4

Calculate: $\int\sin^4 x\,dx$.

Solution:

- $$\int\sin^4 x\,dx=\int\left(\frac{1-\cos 2x}{2}\right)^2 dx=\frac{1}{4}\int 1+\cos^2 2x-2\cos 2x\,dx=\frac{1}{4}\int 1+\frac{1+\cos 4x}{2}-2\cos 2x\,dx=$$
$$=\frac{1}{4}\int\frac{3}{2}+\frac{\cos 4x}{2}-2\cos 2x\,dx=\frac{1}{4}\left(\frac{3}{2}x+\frac{\sin 4x}{8}-\sin 2x\right)+C=\frac{3}{8}x+\frac{\sin 4x}{32}-\frac{\sin 2x}{4}+C$$

Problem 15.5

Calculate: $\int e^x\sin x\,dx$

Solution:

- $\int e^x\sin x\,dx=-e^x\cos x+\int e^x\cos x dx=-e^x\cos x+e^x\sin x-\int e^x\sin x\,dx+C$
- So, $\int e^x\sin x\,dx=\frac{-e^x\cos x+e^x\sin x}{2}+C$

Problem 15.6

Find the equation of the tangent line for the curve $f(x)=x^x$ at $x=1$

Solution:

- $f(x)=x^x=e^{\ln x\cdot x}$
- $f'(x)=\left(e^{\ln x\cdot x}\right)'=(1+\ln x)e^{\ln x\cdot x}$
- $f'(1)=1$, $f(1)=1$
- $y=x$ is the tangent line.

Problem 15.7

Suppose h is a function such that $h'(x)=e^{\sin(x+1)}$ and $h(0)=3$

a. Show that h has an inverse.

b. Find $\left(h^{-1}\right)'(3)$

c. Find $\left(g^{-1}\right)'(3)$, where $g(x)=h(x+1)$

Solution:

a.

- $h'(x)=e^{\sin(x+1)}>0$. So, it is an increasing function $\Rightarrow$ it is one to one function.

b.

- $\left(h^{-1}\right)'(3)=\left(h^{-1}\right)'(h(0))=\dfrac{1}{e^{\sin(0+1)}}=\dfrac{1}{e^{\sin 1}}$

c.

- $\left(g^{-1}\right)'(h(x+1))=\left(g^{-1}\right)'(g(x))=\dfrac{1}{g'(x)}=\dfrac{1}{h'(x+1)}$
- So, $\left(g^{-1}\right)'(g(x))=\left(g^{-1}\right)'(h(x+1))=\dfrac{1}{g'(x)}=\dfrac{1}{h'(x+1)}$
- $\left(g^{-1}\right)'(g(-1))=\left(g^{-1}\right)'(h(0))=\dfrac{1}{g'(-1)}=\dfrac{1}{h'(0)}=\dfrac{1}{e^{\sin 1}}$

Problem 15.8

Suppose f is a continuous function such that $\int_1^x \ln t\cdot f(t)\,dt=\sin\left(x^3+1\right)$. Find $f(x)$.

Solution:

- $\dfrac{d}{dx}\left(\int_1^x \ln t\cdot f(t)\,dt\right)=\dfrac{d}{dx}\left(\sin\left(x^3+1\right)\right)$
- $\ln x\cdot f(x)=3x^2\cdot\cos\left(x^3+1\right)$
- $f(x)=\dfrac{3x^2\cdot\cos\left(x^3+1\right)}{\ln x}$

Problem 15.9

Let $f(x)=|x-3|$.

a. Evaluate $\int_{-2}^{6} f(x)\,dx$

b. Suppose the partition $P=\{-2,0,4,6\}$. Find the upper sum of f for P.

c. Do there exist partitions of $[-2;6]$, where $L_f(P)=0$? Provide a list of all possible partitions, or prove that none exist.

<u>Solution:</u>

a.

- $\int_{-2}^{6}|x-3|\,dx=\int_{-2}^{3}|x-3|\,dx+\int_{3}^{6}|x-3|\,dx=\int_{-2}^{3}x-3\ dx-\int_{3}^{6}x-3\ dx=\frac{x^2}{2}-3x\Big|_{-2}^{3}-\left[\frac{x^2}{2}-3x\Big|_{3}^{6}\right]=17$

b.

- $U_f(P)=\sum_{i=1}^{3}M_i\Delta x_i=5\cdot2+3\cdot4+3\cdot2=28$

c.

- $f(x)\geq0$. So, in order to make the lower sum to be equal to zero, every interval must contain a point the function is exactly equal to zero.
- So, the only possible partitions are $\{-2,6\}$, $\{-2,3,6\}$.