Tutorial 16

Problem 16.1

Give the n-th term of the sequence: $a_1 = 1;\ a_2 = 3;\ a_{n+1} = 2a_n - a_{n-1},\ n \ge 1$

Solution:

- It is easy to see that $a_n = 2n-1$

Problem 16.2

Give the n-th term of the sequence: $a_1 = 1;\ a_{n+1} = a_n + a_{n-1} + \ldots + a_1$

Solution:

- It is easy to see that $a_n = 2^{n-2}$

Problem 16.3

Use mathematical induction: $a_1 = 1;\ a_{n+1} = \frac{n+1}{2n} a_n$, then $a_n = \frac{n}{2^{n-1}}$.

Solution:

- Basis: $a_1 = \frac{1}{2^1 - 1} = 1$
- Assumption: $a_k = \frac{k}{2^k - 1}$
- Prove that: $a_{k+1} = \frac{k+1}{2^{k+1} - 1}$
  - $a_{k+1} = \frac{k+1}{2k} a_k = \frac{k+1}{2k} \frac{k}{2^{k-1}} = \frac{k+1}{2^k}$

Problem 16.4

Let f be a function continuous everywhere and let r be a real number. Define a sequence as follows:

$a_1 = r,\ a_2 = f(r),\ a_3 = f(f(r)),\ldots$

Prove if $a_n \to L$ then L is a fixed point of f: $f(L) = L$.

Given that the limit exists, find it: $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$

Solution:

- $f(L) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n = L$
- $\sqrt{2+L} = L \Rightarrow L = 2$

Problem 16.5

State whether the sequence $\left(\frac{n+1}{n+2}\right)^n$ converges as $n \to \infty$; if it does, find the limit.

Solution:

- $\lim\limits_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n = \lim\limits_{n\to\infty}\left(1-\frac{1}{n+2}\right)^n = e^{\lim_{n\to\infty} -\frac{n}{n+2}} = e^{-1}$

Problem 16.6

State whether the sequence $\int_{-\frac{1}{n}}^{\frac{1}{n}} \sin x^2\, dx$ converges as $n \to \infty$; if it does, find the limit.

Solution:

- $\left|\int_{-\frac{1}{n}}^{\frac{1}{n}} \sin x^2\, dx\right| \le \int_{-\frac{1}{n}}^{\frac{1}{n}} \left|\sin x^2\right| dx \le \int_{-\frac{1}{n}}^{\frac{1}{n}} 1\, dx = \frac{2}{n} \to 0$

Problem 16.7

Express the decimal fraction $0.a_1a_2...a_n$ in sigma notation, using powers of $\frac{1}{10}$.

Solution:

- $0.a_1a_2...a_n = \frac{a_1}{10} + \frac{a_2}{10^2} + ... + \frac{a_n}{10^n} = \sum_{k=1}^{n} \frac{a_k}{10^k}$

Problem 16.8

Show that $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$ diverges, although $\ln\left(\frac{k+1}{k}\right) \to 0$

Solution:

- $s_n = \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n} \ln(k+1) - \ln(k) = \ln(n+1) - \ln 1 = \ln(n+1) \to \infty$

Problem 16.9

Determine whether the series $\sum \frac{2k}{(2k)!}$ converges or diverges.

Solution:

- $\sum \frac{2k}{(2k)!} = \sum \frac{1}{1\cdot 2\cdot ...\cdot(2k-1)} < \sum \frac{1}{k^2}$ - finite.