Tutorial 17

Problem 17.1

Determine whether the series converges or diverges.

$$\frac{1}{4}+\frac{1\cdot 3}{4\cdot 7}+\frac{1\cdot 3\cdot 5}{4\cdot 7\cdot 10}+\frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 7\cdot 10\cdot 13}+...$$

Solution:

- $\frac{1}{4}+\frac{1\cdot 3}{4\cdot 7}+\frac{1\cdot 3\cdot 5}{4\cdot 7\cdot 10}+\frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 7\cdot 10\cdot 13}+...=\sum_{k=0}^{\infty}\frac{1\cdot 3\cdot ...\cdot(2k+1)}{4\cdot 7\cdot ...\cdot(3k+4)}$
- Ratio test: $\frac{1\cdot 3\cdot ...\cdot(2k+3)}{4\cdot 7\cdot ...\cdot(3k+7)}/\frac{1\cdot 3\cdot ...\cdot(2k+1)}{4\cdot 7\cdot ...\cdot(3k+4)}=\frac{(2k+3)}{(3k+7)}\to\frac{2}{3}$
- That is why the series is convergent.

Problem 17.2

Determine whether the series converges or diverges: $\sum\frac{2+\sin k}{k^2}$

Solution:

- $\sum\frac{2+\sin k}{k^2}$ is increasing, because $\frac{2+\sin k}{k^2}$ is positive.
- $\sum\frac{2+\sin k}{k^2}\le\sum\frac{2+1}{k^2}=3\sum\frac{1}{k^2}<3\sum\frac{1}{k(k+1)}\to 3$
- So, the series converges.

Problem 17.3

Find the integers $p\ge 2$ for which $\sum\frac{(k!)^2}{(pk)!}$ converges.

Solution:

- The ratio test: $\frac{((k+1)!)^2}{(p(k+1))!}/\frac{(k!)^2}{(pk)!}=\frac{(k+1)^2}{(pk+1)(pk+2)\cdot ...\cdot(pk+p)}\to\begin{cases}\frac{1}{4}, & if\ \ p=2\\ 0, & if\ \ p=0\end{cases}$
- The series is convergent.

Problem 17.4

Take $r>0$ and let the a_k be positive. Use the root test to show that, if $(a_k)^{\frac{1}{k}}\to\rho,\ \rho<\frac{1}{r}$ then $\sum a_k r^k$ converges.

Solution:

- $\sqrt[k]{a_k r^k} \to r \cdot \rho < r \cdot \frac{1}{r} = 1$ - the series is convergent.

Problem 17.5

Show that if $\sum a_k$ is absolutely convergent then $\sum a_k^2$ is convergent.

Solution:

- Given that $\sum |a_k|$ converges. So $|a_k| \to 0$
- So, exists N such that $\sum_{k=N}^{\infty} a_k^2 < \sum_{k=N}^{\infty} |a_k| < \infty$
- That is why $\sum_{k=N}^{\infty} a_k^2$ converges $\Rightarrow \sum a_k^2$ converges.

Problem 17.6

Show an example that the claim: "if $\sum a_k^2$ is convergent then $\sum a_k$ is absolutely convergent" is wrong.

- $\sum \frac{1}{k^2}$ converges, but $\sum (-1)^k \frac{1}{k}$ is not absolutely convergent.

Problem 17.7

Test the series for absolute and conditional convergence: $\frac{2 \cdot 3}{4 \cdot 5} - \frac{5 \cdot 6}{7 \cdot 8} + ... + (-1)^k \frac{(3k+2)\cdot(3k+3)}{(3k+4)\cdot(3k+5)} + ...$

Solution:

- $\lim_{k \to \infty} \frac{(3k+2)\cdot(3k+3)}{(3k+4)\cdot(3k+5)} = 1$ - the sequence diverges.

Problem 17.8

Expand $x \ln x$ in powers of $x-2$.

Solution:

- $g(2) = 2\ln 2,\ g'(2) = 1 + \ln 2,\ g^{(k)}(2) = \frac{(-1)^k (k-2)!}{2^{k-1}},\ k \ge 2$
- $g(x) = 2\ln 2 + (1+\ln 2)(x-2) + \sum_{k=2}^{\infty} \frac{(-1)^k (k-2)!}{2^{k-1} \cdot k!} (x-2)^k$

Problem 17.9

Expand $(x-1)^n$ in powers of x.

Solution:

- $(x-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$