Tutorial 18

Problem 18.1

Expand $x\ln x$ in powers of $x-2$.

Solution:

- $g(2)=2\ln 2,\ g'(2)=1+\ln 2,\ g^{(k)}(2)=\dfrac{(-1)^k(k-2)!}{2^{k-1}},\ k\ge 2$
- $g(x)=2\ln 2+(1+\ln 2)(x-2)+\sum_{k=2}^{\infty}\dfrac{(-1)^k(k-2)!}{2^{k-1}\cdot k!}(x-2)^k$

Problem 18.2

Expand $x\cdot e^x$ in powers of x.

Solution:

- $e^x=\sum_{i=0}^{\infty}\dfrac{x^i}{i!}$
- $x\cdot e^x=\sum_{i=0}^{\infty}\dfrac{x^{i+1}}{i!}$

Problem 18.3

Assume that f is a function with $\left|f^{(n)}(x)\right|\le 1$ for all n and all real x.

Find the least integer n for which you can be sure that $P_n(2)$ approximates $f(2)$ within 0.001.

Solution:

- $\left|f(2)-P_n(2)\right|=\left|R_n(2)\right|\le(1)\dfrac{2^{n+1}}{(n+1)!}=\dfrac{2^{n+1}}{(n+1)!}$
- $\dfrac{2^{n+1}}{(n+1)!}<0.001\Rightarrow$ the least integer is $n=9$

Problem 18.4

For $f(x)=e^x$ find the Tailor polynomial P_n that approximates e^x at $x=\dfrac{1}{2}$ with 4 decimal place accuracy.

Then use that polynomial to obtain an estimate of $\sqrt{e}$.

Solution:

- We want $\left|R_n\left(\dfrac{1}{2}\right)\right|<0.00005$, so $0<c<\dfrac{1}{2}$

- We have $\left|R_n\left(\frac{1}{2}\right)\right| = \frac{e^c}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} < \frac{e^{\frac{1}{2}}}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} < \frac{2}{(n+1)!}\left(\frac{1}{2}\right)^{n+1}$
- $\frac{2}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} < 0.00005$
- So, it is enough to take $n \geq 5$
- $P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$
- $P_5(0.5) = 1 + 0.5 + \frac{0.5^2}{2!} + \frac{0.5^3}{3!} + \frac{0.5^4}{4!} + \frac{0.5^5}{5!} \approx 1.6487$

Problem 18.5

Find the Tailor polynomial P_n for f of least degree that approximates e^x at $x = -1$ with 3 decimal place accuracy. Then use that polynomial to obtain an estimate of $\frac{1}{e}$

Solution:

- We want $\left|R_n(-1)\right| < 0.0005$, so $-1 < c < 0$
- We have $\left|R_n(-1)\right| = \frac{e^c}{(n+1)!}\left(\left|-1\right|\right)^{n+1} < \frac{1}{(n+1)!}$
- $\frac{1}{(n+1)!} < 0.0005$
- So, it is enough to take $n \geq 7$
- $P_7(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$
- $P_7(-1) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} \approx 0.368$

Problem 18.6

Expand $\sin\frac{\pi x}{2}$ in powers of $x - 1$ and specify the values of x for which the expansion is valid.

Solution:

- $$\sin\frac{\pi x}{2} = \sin\left[\frac{\pi}{2}(x-1) + \frac{\pi}{2}\right] = \sin\left[\frac{\pi}{2}(x-1)\right]\cos\frac{\pi}{2} + \cos\left[\frac{\pi}{2}(x-1)\right]\sin\frac{\pi}{2} =$$
  $$= \cos\left[\frac{\pi}{2}(x-1)\right] = \sum_{k=0}^{\infty}(-1)^k\left(\frac{\pi}{2}\right)^{2k}\frac{(x-1)^{2k}}{(2k)!}$$

Problem 18.7

Expand $\ln(1+2x)$ in powers of $x-1$ and specify the values of x for which the expansion is valid.

Solution:

- $$\ln(1+2x)=\ln\left(3+2(x-1)\right)=\ln\left[3\left(1+\frac{2}{3}(x-1)\right)\right]=\ln 3+\ln\left[1+\frac{2}{3}(x-1)\right]=$$ $$=\ln 3+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{2}{3}\right)^k(x-1)^k$$

Problem 18.8

Expand $2x^5+x^2-3x-5$ in powers of $x+1$ and specify the values of x for which the expansion is valid.

Solution:

- $2x^5+x^2-3x-5=-3+5(x+1)-19(x+1)^2+20(x+1)^3-10(x+1)^4+2(x+1)^5$