Tutorial 3

Problem 3.1

Prove that for any positive integer number n, n^3+2n is divisible by 3.

Solution:

Basis: $n=1$: $n^3+2n=1+2=3$ is divisible by 3.

Assumption: $n=k$: k^3+2k is divisible by 3.

We have to prove: $n=k+1$: $(k+1)^3+2(k+1)$ is divisible by 3.

$(k+1)^3+2(k+1)=k^3+3k^2+3k+1+2k+2=(k^3+2k)+3(k^2+k+1)$ and is divisible by 3, because

(k^3+2k) is divisible by 3 by the assumption and $3(k^2+k+1)$ is divisible by 3.

Problem 3.2

Prove that $3^n>n^2$ for $n=1, n=2$ and use the mathematical induction to prove that $3^n>n^2$ for n a positive integer greater than 2.

Solution:

Basis: $n=1,2$: $3^1>1^2$, $3^2>2^2$

Assumption: $n=k$: $3^k>k^2$

We have to prove: $n=k+1$: $3^{k+1}>(k+1)^2$

$3^{k+1}=3\cdot 3^k>3\cdot k^2=k^2+k^2+k^2>k^2+2k+1=(k+1)^2$

Problem 3.3

Prove that $n!>2^n$ for n a positive integer greater than or equal to 4.

Solution:

Basis: $n=4$: $4!>2^4$

Assumption: $k!>2^k$

We have to prove: $(k+1)!>2^{k+1}$

$(k+1)!=(k+1)\cdot k!>(k+1)\cdot 2^k>2^{k+1}$

Problem 3.4

Find algebraically: $\lim\limits_{x\to -1}\dfrac{x^3+2x^2+3x+2}{x^3-x^2-x+1}$

Solution:

$$\lim_{x\to -1}\frac{x^3+2x^2+3x+2}{x^3-x^2-x+1}=\lim_{x\to -1}\frac{(x+1)(x^2+x+2)}{x^2(x-1)-(x-1)}=\lim_{x\to -1}\frac{(x+1)(x^2+x+2)}{(x-1)(x^2-1)}=$$

$$=\lim_{x\to -1}\frac{(x+1)(x^2+x+2)}{(x-1)^2(x+1)}=\lim_{x\to -1}\frac{(x^2+x+2)}{(x-1)^2}=\frac{1}{2}$$

Problem 3.5

Find algebraically: $\lim_{x\to 4}\frac{x^2-16}{\sqrt{3x+4}-4}$

Solution:

$$\lim_{x\to 4}\frac{x^2-16}{\sqrt{3x+4}-4}=\lim_{x\to 4}\frac{(x^2-16)(\sqrt{3x+4}+4)}{(3x+4)-16}=\lim_{x\to 4}\frac{(x-4)(x+4)(\sqrt{3x+4}+4)}{3(x-4)}=\lim_{x\to 4}\frac{(x+4)(\sqrt{3x+4}+4)}{3}=\frac{64}{3}$$

Problem 3.6

How close to 5 we have to take $x+3$ so that so that $5x+8$ is within a distance of 0.1 from 18?

Solution:

The distance from $x+3$ to 5 is equivalent to the distance from x to 2.

$$|(5x+8)-18|=|5x-10|=5|x-2|<0.1 \Rightarrow |x-2|<0.02$$

Problem 3.7

Prove directly using the formal definition: $\lim_{x\to 4} ax+b$, when $a,b\in R$.

Solution:

Precise definition:

For each real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all x with $|x - c| < \delta$, we have $|f(x) - L| < \varepsilon$,

If we choose $L=4a+b$ then we get:

for each $\varepsilon>0$ we have to find a real $\delta>0$ such that for all x with $|x-4|<\delta$, we have

$|ax+b-(4a+b)|=|a(x-4)|<a\delta$. So, it is enough to choose $\delta=\frac{\varepsilon}{a}$.

Problem 3.8

Prove directly using the formal definition: $\lim_{x\to 1}\frac{5x+1}{x^2+x+1}=2$

Solution:

For each $\varepsilon>0$ we have to find a real $\delta>0$ such that for all x with $|x-1|<\delta$, we would have

$\left|\frac{5x+1}{x^2+x+1}-2\right|<\varepsilon$. We assume $\delta<0.1$.

$$\left|\frac{5x+1}{x^2+x+1}-2\right|=\left|\frac{5x+1-2\left(x^2+x+1\right)}{x^2+x+1}\right|=\left|\frac{2x^2-3x+1}{x^2+x+1}\right|=\left|\frac{2\left(x-0.5\right)}{x^2+x+1}\right||x-1|<\left|\frac{2}{0^2+0+1}\right||x-1|<2\delta .$$

So, it is enough to choose $\delta=\min\left\{\frac{\varepsilon}{2};0.1\right\}$.

Problem 3.9

Prove directly using the formal definition: $\lim_{x\to1}\sqrt[3]{x+7}=2$

Solution:

For each $\varepsilon>0$ we have to find a real $\delta>0$ such that for all x with $|x-1|<\delta$, we would have $\left|\sqrt[3]{x+7}-2\right|<\varepsilon$. We assume $\delta<0.1$.

$$\left|\sqrt[3]{x+7}-2\right|=\left|\frac{x+7-8}{\sqrt[3]{\left(x+7\right)^2}+\sqrt[3]{\left(x+7\right)}+4}\right|=\left|\frac{1}{\sqrt[3]{\left(x+7\right)^2}+\sqrt[3]{\left(x+7\right)}+4}\right||x-1|\le\left|\frac{1}{\sqrt[3]{\left(-6+7\right)^2}+\sqrt[3]{\left(-6+7\right)}+4}\right||x-1|=$$

$$=\frac{1}{6}|x-1|<\frac{1}{6}\delta$$

So, it is enough to choose $\delta=\min\{6\varepsilon;0.1\}$.

Problem 3.10

Prove directly using the formal definition: $\lim_{x\to2}\frac{x-1}{x+1}=\frac{1}{3}$

Solution:

For each $\varepsilon>0$ we have to find a real $\delta>0$ such that for all x with $|x-1|<\delta$, we would have $\left|\frac{x-1}{x+1}-\frac{1}{3}\right|<\varepsilon$. We assume $\delta<0.1$.

$$\left|\frac{x-1}{x+1}-\frac{1}{3}\right|=\left|\frac{3x-3-x-1}{3\left(x+1\right)}\right|=\left|\frac{2}{3\left(x+1\right)}\right||x-1|\le\left|\frac{2}{3(1+1)}\right||x-1|=\frac{1}{3}|x-1|<\frac{1}{3}\delta$$

So, it is enough to take $\delta=\min\{3\varepsilon;0.1\}$.