Tutorial 4

Problem 4.1

Show that the following limit doesn't exist, using a definition $\lim_{x\to 0}\frac{1}{|x|}$.

Solution:

For each $L\in R$, we must show that exists $\varepsilon>0$ such that for any $\delta>0$, $0<|x|<\delta$ does not imply that $\left|\frac{1}{x}-L\right|<\varepsilon$.

It is obvious that $L\geq 0$.

For any $\delta>0$, any $x<\frac{1}{L+2\varepsilon}$ exists $\left|\frac{1}{x}-L\right|=\left|\frac{1}{\frac{1}{L+2\varepsilon}}-L\right|=|2\varepsilon|=2\varepsilon>\varepsilon$.

Problem 4.2

Show that the following limit doesn't exist: $\lim_{x\to 2} f(x)$, when $f(x)=\begin{cases}3+x^2, & if\ \ x\geq 2\\ 4-x-x^3, & if\ \ x<2\end{cases}$

Solution *A*:

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} 3+x^2=7$, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 4-x-x^3=-6$

$\lim_{x\to 2^+} f(x)\neq\lim_{x\to 2^-} f(x)$, that is why the limit doesn't exist.

Solution B:

Case 1: $L=7$

We must show that exists $\varepsilon>0$ such that for any $\delta>0$, $0<|x|<\delta$ does not imply that $|f(x)-7|<\varepsilon$.

We have to show that for any $\delta>0$ exist $0<|x|<\delta$ such that $|f(x)-7|>\varepsilon$.

For $x<2$ we get: $|f(x)-7|=|4-x-x^3-7|=|x^3+x+3|$. For $\varepsilon=1$ we get that for any $x>0$ $|x^3+x+3|>\varepsilon$.

Case 2: $L\neq 7$

We can sign $L=7+A$, when $A>0$.

We must show that exists $\varepsilon>0$ such that for any $\delta>0$, $0<|x|<\delta$ does not imply that $|f(x)-7|<\varepsilon$.

We have to show that for any $\delta>0$ exist $0<|x|<\delta$ such that $|f(x)-7|>\varepsilon$.

For $x\geq 2$ we get: $|f(x)-(7+A)|=|3+x^2-(7+A)|=|x^2-4-A|>|A|-|x^2-4|$.

For $\varepsilon = \frac{|A|}{2}$ we get that for any $\sqrt{4-\frac{|A|}{2}} < x < \sqrt{4+\frac{|A|}{2}}$ exists $|A| - |x^2 - 4| > |A| - \frac{|A|}{2} = \frac{|A|}{2}$. So,

$|f(x) - (7+A)| > \varepsilon$.

Problem 4.3

Does the following limit exist $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

Solution:

We will prove that $\lim_{x\to 0}\left|x\sin\left(\frac{1}{x}\right)\right| = 0$. Then $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

Using the pinching theorem (theorem 2.5.1) we obtain: $0 \le \lim_{x\to 0}\left|x\sin\left(\frac{1}{x}\right)\right| \le \lim_{x\to 0}|x|\left|\sin\left(\frac{1}{x}\right)\right| \le \lim_{x\to 0}|x| = 0$

Problem 4.4

Prove that there is some number $x \in R$ such that $x^5 + (\sin x)^{\cos x} = 1 - 5\cos x$

Solution:

$x^5 + (\sin x)^{\cos x} = 1 - 5\cos x \iff x^5 + (\sin x)^{\cos x} + 5\cos x - 1 = 0$

We are going to use the intermediate value theorem.

$(\sin x)^{\cos x} + 5\cos x - 1$ is bounded, because $(\sin x)^{\cos x}$ and $5\cos x - 1$ are bounded.

$\lim_{x\to+\infty} x^5 = +\infty$, $\lim_{x\to-\infty} x^5 = -\infty$. So, using the intermediate value theorem, we obtain that there is some number $x \in R$ such that $x^5 + (\sin x)^{\cos x} + 5\cos x - 1 = 0$

Problem 4.5

Prove that there is some number $x \in R$ such that $x^9 + e^x = 8$

Solution:

We are going to use the intermediate value theorem.

It is obvious that $\lim_{x\to+\infty}\left(x^9 + e^x\right) = +\infty$.

$\lim_{x\to-\infty} x^9 = -\infty$, $\lim_{x\to-\infty} e^x = 0 \Rightarrow \lim_{x\to-\infty}\left(x^9 + e^x\right) = -\infty$

So, using the intermediate value theorem, we obtain that there is some number $x \in R$ such that $x^9 + e^x = 8$.

Problem 4.6

Prove or give a counterexample: "If $f^2(x)$ is continuous then $f(x)$ is continuous".

Solution:

The counterexample is $f(x)=\begin{cases}1, & x\in Q\\ -1, & x\notin Q\end{cases}$

It is easy to see that $f^2(x)=1$ is continuous, but $f(x)$ is not.

Problem 4.7

Prove or give a counterexample: "If $\sin\left(f(x)\right)$ is continuous then $f(x)$ is continuous".

Solution:

The counterexample is $f(x)=2\pi\lfloor x\rfloor$.

It is easy to see that $\sin\left[f(x)\right]=0$, but $f(x)$ is not continuous.

Problem 4.8

Evaluate or show the limit doesn't exist: $\lim\limits_{x\to 0}\dfrac{\sin x}{\tan 5x}$

Solution:

$$\lim_{x\to 0}\frac{\sin x}{\tan 5x}=\lim_{x\to 0}\frac{\sin x}{\frac{\sin 5x}{\cos 5x}}=\lim_{x\to 0}\frac{\sin x}{\frac{\sin 5x}{1}}=\lim_{x\to 0}\frac{\sin x}{\sin 5x}=\frac{1}{5}\lim_{x\to 0}\frac{\sin x}{x}\frac{5x}{\sin 5x}=\frac{1}{5}$$

Problem 4.9

Evaluate or show the limit doesn't exist: $\lim\limits_{x\to 0}\dfrac{\sin x}{|\sin x|}$

Solution:

$$\lim_{x\to 0^-}\frac{\sin x}{|\sin x|}=\lim_{x\to 0^-}\frac{\sin x}{-\sin x}=-1,\quad \lim_{x\to 0^+}\frac{\sin x}{|\sin x|}=\lim_{x\to 0^-}\frac{\sin x}{\sin x}=1$$

$$\lim_{x\to 0^+}\frac{\sin x}{|\sin x|}\neq\lim_{x\to 0^-}\frac{\sin x}{|\sin x|}\text{, so }\lim_{x\to 0}\frac{\sin x}{|\sin x|}\text{ doesn't exist.}$$