Tutorial 5

Problem 5.1

Differentiate the following function: $f\left(x\right)=\ln\frac{\sin x}{\cos x+\sqrt{\cos 2x}}$

Solution:

$$f'\left(x\right)=\left(\ln\frac{\sin x}{\cos x+\sqrt{\cos 2x}}\right)'=\frac{\cos x+\sqrt{\cos 2x}}{\sin x}\left(\frac{\sin x}{\cos x+\sqrt{\cos 2x}}\right)'=$$

$$=\frac{\cos x+\sqrt{\cos 2x}}{\sin x}\frac{\cos x\left(\cos x+\sqrt{\cos 2x}\right)-\sin x\left(\cos x+\sqrt{\cos 2x}\right)'}{\left(\cos x+\sqrt{\cos 2x}\right)^2}=$$

$$=\frac{\cos x+\sqrt{\cos 2x}}{\sin x}\frac{\cos x\left(\cos x+\sqrt{\cos 2x}\right)-\sin x\left(-\sin x+\frac{-2\sin 2x}{2\sqrt{\cos 2x}}\right)'}{\left(\cos x+\sqrt{\cos 2x}\right)^2}=$$

$$=\frac{1}{\sin x}\frac{\cos^2 x+\sin^2 x+\frac{1}{\sqrt{\cos 2x}}\left(\cos x\cos 2x+\sin x\cdot\sin 2x\right)}{\left(\cos x+\sqrt{\cos 2x}\right)}=$$

$$=\frac{1}{\sin x}\frac{1+\frac{\cos x}{\sqrt{\cos 2x}}}{\left(\cos x+\sqrt{\cos 2x}\right)}=\frac{1}{\sin x\sqrt{\cos 2x}}$$

Problem 5.2

Differentiate the following function: $f\left(x\right)=x^2\sin\left(3x\right)\cos\left(\sqrt{x}+1\right)$

Solution:

$$f'\left(x\right)=\left(x^2\sin\left(3x\right)\cos\left(\sqrt{x}+1\right)\right)'=\left(x^2\sin\left(3x\right)\right)'\cos\left(\sqrt{x}+1\right)-\left(x^2\sin\left(3x\right)\right)\sin\left(\sqrt{x}+1\right)\frac{1}{2\sqrt{x}}=$$

$$=\left(2x\sin\left(3x\right)+3x^2\cos\left(3x\right)\right)\cos\left(\sqrt{x}+1\right)-\left(x^2\sin\left(3x\right)\right)\sin\left(\sqrt{x}+1\right)\frac{1}{2\sqrt{x}}$$

Problem 5.3

Prove that the function $f\left(x\right)=\begin{cases}x\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$ is not differentiable at 0.

Solution:

$f'\left(0\right)=\lim\limits_{h\to 0}\frac{f\left(0+h\right)-f\left(0\right)}{h}=\lim\limits_{h\to 0}\frac{h\sin\frac{1}{h}-0}{h}=\lim\limits_{h\to 0}\sin\frac{1}{h}$. The limit doesn't exist.

Problem 5.4

Find $f'(5)$, if exists, when $f(x)=\begin{cases}30x-67, & x<5\\ 3x^2+8, & x\geq 5\end{cases}$

Solution:

$$\lim_{h\to 0^+}\frac{f(5+h)-f(5)}{h}=\lim_{h\to 0^+}\frac{3(5+h)^2+8-83}{h}=\lim_{h\to 0^+}\frac{30h+3h^2}{h}=30$$

$$\lim_{h\to 0^-}\frac{f(5+h)-f(5)}{h}=\lim_{h\to 0^+}\frac{30(5+h)-30\cdot 5}{h}=30$$

That is why $f'(5)=30$.

Problem 5.5

Find the area of the triangle formed by the x axis and the lines tangent and normal to the graph of $f(x)=x^2-5$ at the point $(3;4)$.

Solution:

The tangent line at the point $(3;4)$ is $y-4=f'(3)(x-3)\ \Rightarrow\ y-4=6(x-3)\ \Rightarrow\ y=6x-14$

The line intersects the x-axis at $0=6x-14\ \Rightarrow x=\frac{7}{3}$

The normal line at the point $(3;4)$ is $y-4=-\frac{1}{6}(x-3)\ \Rightarrow\ y=-\frac{1}{6}x+4.5$

The line intersects the x-axis at $0=-\frac{1}{6}x+4.5\ \Rightarrow\ \ x=27$

So, the area of the triangle is $\frac{1}{2}\cdot 4\cdot\left(27-\frac{7}{3}\right)=\frac{88}{3}$

Problem 5.6

Find the derivative: $\frac{d^{100}}{dx^{100}}\left(\sum_{i=0}^{99}a_i x^i\right)$

Solution:

$$\frac{d^{100}}{dx^{100}}\left(\sum_{i=0}^{99}a_i x^i\right)=0$$

Problem 5.7

Find the rate of change of the surface of a triangle as a function of an angle between two given sides.

Solution:

The surface formula is $S = \frac{1}{2}ab\sin\alpha$, when a,b are the sides and α is an angle between them.

$$\frac{dS}{d\alpha} = \frac{1}{2}ab\cos\alpha$$

Problem 5.8

Find the 100th derivative of $f(x) = \frac{6x+10}{(x+3)(x+1)}$

Solution:

We can present: $\frac{6x+10}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$

$$\frac{A}{x+3} + \frac{B}{x+1} = \frac{(A+B)x + 3B + A}{(x+3)(x+1)}$$

So, $\begin{cases} A+B=6 \\ 3B+A=10 \end{cases} \Rightarrow \begin{cases} B=6-A \\ 3(6-A)+A=10 \end{cases} \Rightarrow \begin{cases} B=6-A \\ 2A=8 \end{cases} \Rightarrow \begin{cases} B=2 \\ A=4 \end{cases}$

$$\frac{6x+10}{(x+3)(x+1)} = \frac{2}{x+3} + \frac{4}{x+1}$$

$$\left(\frac{2}{x+3} + \frac{4}{x+1}\right)^{(100)} = \left(\frac{2}{x+3}\right)^{(100)} + \left(\frac{4}{x+1}\right)^{(100)} = (-1)^{100}\frac{2\cdot 100!}{(x+3)^{101}} + (-1)^{100}\frac{4\cdot 100!}{(x+1)^{101}} = \frac{2\cdot 100!}{(x+3)^{101}} + \frac{4\cdot 100!}{(x+1)^{101}}$$