Tutorial 7

Problem 7.1

The cost of erecting a small building is $1,000,000 for the first story, $1,100,000 for the second, $1,200,000 for the third and so on. Other expenses (lot, basement, etc.) are $5,000,000. Assume that the annual rent is $200,000 per story. How many stories will provide the greatest return on investment?

Solution:

Let x is a number of stories.

The total expenses in millions are: $5+k\cdot 1+\left(0+0.1+...+0.1\cdot(k-1)\right)=5+k+0.1\frac{k-1}{2}k=0.05k^2+0.95k+5$

The annual rent is $0.2k$.

The return is $\frac{0.2k}{0.05k^2+0.95k+5}$ and it should be maximal.

$$\left(\frac{0.2k}{0.05k^2+0.95k+5}\right)'=\left(\frac{0.2\left(0.05k^2+0.95k+5\right)-0.2k\left(0.1k+0.95\right)}{\left(0.05k^2+0.95k+5\right)^2}\right)'=$$

$$=\frac{0.2\left(0.05k^2+0.95k+5\right)-0.02k^2-0.19k}{\left(0.05k^2+0.95k+5\right)^2}=\frac{1-0.01k^2}{\left(0.05k^2+0.95k+5\right)^2}=0$$

$\Rightarrow$ $k=10$ - it is easy to check that it is a maximum point

Problem 7.2

The distance from a point to a line is the distance from that point to the closest point of the line. What point of the line $Ax+By+C=0$ ($B\neq 0$) is closest to the point $(x_1;y_1)$.

Solution:

$Ax+By+C=0 \Leftrightarrow y=-\frac{C}{B}-\frac{A}{B}x$. So every point of the line is $\left(x,-\frac{C}{B}-\frac{A}{B}x\right)$

So, the distance from the point $(x_1;y_1)$ to the line is $\sqrt{(x_1-x)^2+\left(y_1+\frac{C}{B}+\frac{A}{B}x\right)^2}$ and it should be minimal.

If $\sqrt{(x_1-x)^2+\left(y_1+\frac{C}{B}+\frac{A}{B}x\right)^2}$ is minimal then $(x_1-x)^2+\left(y_1+\frac{C}{B}+\frac{A}{B}x\right)^2$ is also minimal.

$$\left((x_1-x)^2+\left(y_1+\frac{C}{B}+\frac{A}{B}x\right)^2\right)'=2(x-x_1)+2\left(y_1+\frac{C}{B}+\frac{A}{B}x\right)=0 \Rightarrow x=\frac{(x_1-y_1)-\frac{C}{B}}{1+\frac{A}{B}}, \text{ and } y=-\frac{C}{B}-\frac{A}{B}x.$$

Problem 7.3

Sketch the graph of the function $f(x)=x^2(x-7)^{\frac{1}{3}}$.

Solution:

1. Domain

   Any $x\in R$.

2. Intercept

   y: $f(0)=0$.

   x: $f(x)=0 \Leftrightarrow x^2(x-7)^{\frac{1}{3}}=0 \Leftrightarrow x=0$ or $x=7$.

We have two points of intersection: $(0;0),(7;0)$

3. Symmetry/periodicity.

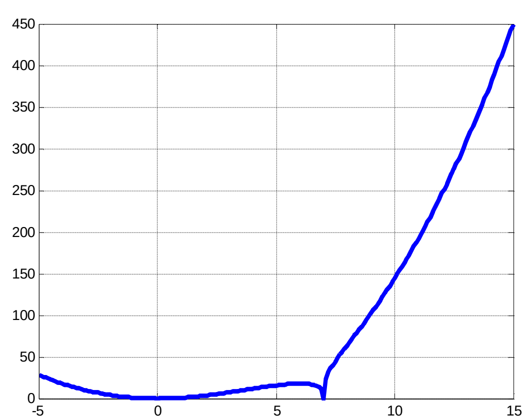
   No symmetry or periodicity.

4. First derivative.

- $f'(x)=2x(x-7)^{\frac{1}{3}}+x^2\frac{1}{3}(x-7)^{-\frac{2}{3}}$
- $2x(x-7)^{\frac{1}{3}}+x^2\frac{1}{3}(x-7)^{-\frac{2}{3}}=0 \Leftrightarrow \frac{6x(x-7)+x^2}{x-7}=0 \Leftrightarrow x\frac{x-6}{x-7}=0 \Leftrightarrow x=0;6$
- $x\frac{x-6}{x-7}>0 \Leftrightarrow x>7,\ 0<x<6$, $x\frac{x-6}{x-7}<0 \Leftrightarrow x<0,\ 6<x<7$

5. Second derivative.

- $f''(x)=\left(x\frac{x-6}{x-7}\right)'=\frac{x-6}{x-7}+x\frac{(x-7)-(x-6)}{(x-7)^2}=\frac{x-6}{x-7}-\frac{x}{(x-7)^2}=\frac{(x-6)(x-7)-x}{(x-7)^2}=\frac{x^2-14x+42}{(x-7)^2}$
- $\frac{x^2-14x+42}{(x-7)^2}=0 \Leftrightarrow x=7\pm\sqrt{7}$, $\frac{x^2-14x+42}{(x-7)^2}>0 \Leftrightarrow x<7-\sqrt{7}$ or $x>7+\sqrt{7}$



Problem 7.4

Sketch the graph of the function $f(x) = 2\cos x + \sin^2 x$.

Solution:

1. Domain

   Any $x \in R$.

2. Intercept

   y: $f(0) = 2$.

   x: $f(x) = 0 \Leftrightarrow 2\cos x + \sin^2 x = 0 \Leftrightarrow \cos^2 x - 2\cos x - 1 = 0 \Leftrightarrow \cos x = 1 \pm \sqrt{2} \Leftrightarrow \cos x = 1 - \sqrt{2} \Leftrightarrow$

   $x = \pm \arccos\left(1 - \sqrt{2}\right) + 2\pi k,\ k \in Z$

We have infinitely many points of intersection: $(0;2), \left(\pm \arccos\left(1-\sqrt{2}\right) + 2\pi k; 0\right)$

3. Symmetry/periodicity.

   $f(-x) = f(x)$ - even function.

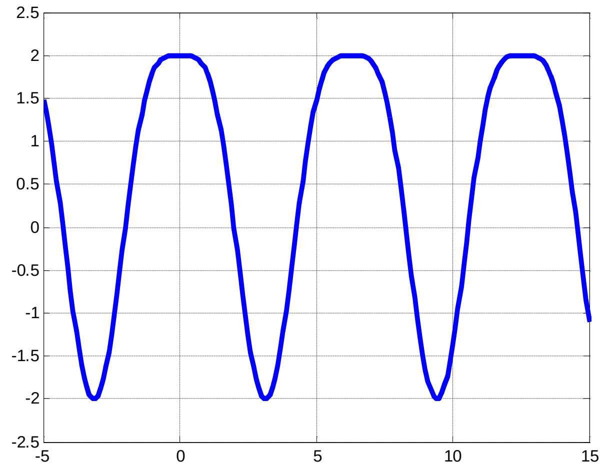
   2π is a period.

4. First derivative.

- $f'(x) = -2\sin x + 2\sin x \cos x = 2\sin x(\cos x - 1)$
- $f'(x) = 2\sin x(\cos x - 1) = 0 \Leftrightarrow \sin x = 0$ or $\cos x = 1 \Leftrightarrow x = \pi k,\ k \in Z$ or $x = 2\pi k,\ k \in Z \Rightarrow$ $x = \pi k,\ k \in Z$
- $f'(x) = 2\sin x(\cos x - 1) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow \pi + 2\pi k < x < 2\pi + 2\pi k,\ k \in Z$
- $f'(x) = 2\sin x(\cos x - 1) < 0 \Leftrightarrow \sin x > 0 \Leftrightarrow 2\pi k < x < \pi + 2\pi k,\ k \in Z$

5. Second derivative.

- $f''(x) = (-2\sin x + 2\sin x \cos x)' = -2\cos x + 2\cos 2x$
- $-2\cos x + 2\cos 2x = 0 \Leftrightarrow 2\cos^2 x - \cos x - 1 = 0 \Leftrightarrow \cos x = 1; -\frac{1}{2} \Leftrightarrow x = 2\pi k;\ \pm\frac{\pi}{3} + 2\pi k,\ k \in Z$
- $2\cos^2 x - \cos x - 1 > 0 \Leftrightarrow (\cos x - 1)\left(\cos x + \frac{1}{2}\right) > 0 \Leftrightarrow \cos x + \frac{1}{2} < 0 \Leftrightarrow \frac{2}{3}\pi + \pi k < x < \frac{4}{3}\pi + \pi k$.
- We can see that concave down interval is two times wider than the concave up interval.



Problem 7.5

Sketch the graph of the function $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$.

Solution:

1. Domain

Any $x\in R$.

2. Intercept

y: $f(0)=0$.

x: $f(x)=0 \Leftrightarrow x\sin\left(\frac{1}{x}\right)=0 \Leftrightarrow x=0$ or $\frac{1}{x}=\pi k,\ k\in Z \Leftrightarrow x=\frac{1}{\pi k}$

We have infinitely many points of intersection: $(0;0),\left(\frac{1}{\pi k};0\right),\ k\in Z$.

3. Symmetry/periodicity.

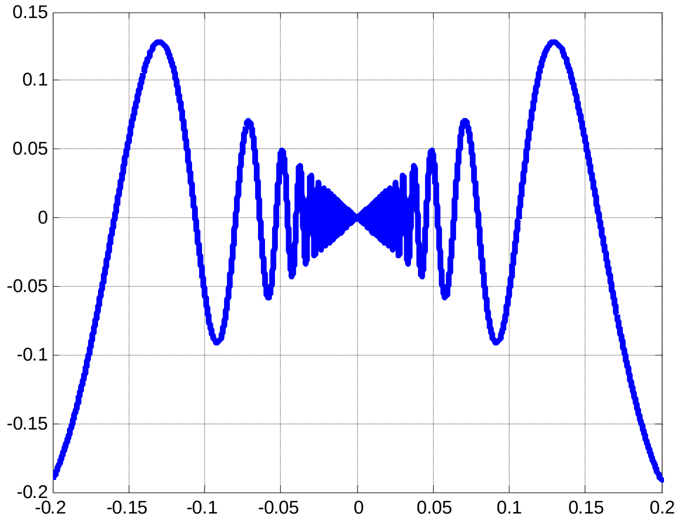
$f(-x)=f(x)$ - even function.

4. First derivative.

- $\left(x\sin\left(\frac{1}{x}\right)\right)'=\sin\left(\frac{1}{x}\right)+x\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)=\sin\left(\frac{1}{x}\right)-\frac{1}{x}\cos\left(\frac{1}{x}\right)=0$

It is hard to solve the above equation. The formal approach doesn't help us too much. That is why it is better to use a general approach as we did in class. Best of all is to sketch graphs in the following order:

1. $f(x)=\sin(x)$ 2. $f(x)=\sin\left(\frac{1}{x}\right)$ 3. $f(x)=x\sin\left(\frac{1}{x}\right)$


0.15
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-0.05
-0.1
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-0.2
-0.2
-0.15
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0
0.05
0.1
0.15
0.2