Tutorial 8

Problem 8.1

Suppose that f and g are differentiable functions and $f(x)g'(x)-g(x)f'(x)$ has no zeros on some interval I. Assume that there are numbers a,b in I with $a<b$ for which $f(a)=f(b)=0$, and that f has no zeros in (a,b). Prove that if $g(a)\neq 0$ and $g(b)\neq 0$, then g has exactly one zero in (a,b).

Solution:

- Assume g has no zeros in $(a;b)$.
    - Consider $h(x)=\dfrac{f(x)}{g(x)}$. $h(a)=h(b)=0$. So, by mean value theorem exists $c\in(a;b)$ such that $h'(c)=0$.
    - Then $h'(x)=\dfrac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}\neq 0$. Contradiction.
- So, g has zeros in $(a;b)$.
- Assume g has two zeros or more in $(a;b)$.
    - Consider $h(x)=\dfrac{g(x)}{f(x)}$. It has two zeros or mote.
    - By mean value theorem $h'(x)$ has at least one zero in $(a;b)$.
    - But $h'(x)=\dfrac{g'(x)f(x)-g(x)f'(x)}{f^2(x)}\neq 0$. Contradiction.
- So, g has exactly one zero in $(a;b)$.

Problem 8.2

Assume that f and g are differentiable on the interval $(-c;c)$ and $f(0)=g(0)$.

a. Show that if $f'(x)>g'(x)$ for all $x\in(0;c)$ then $f(x)>g(x)$ for all $x\in(0;c)$.

b. Show that if $f'(x)>g'(x)$ for all $x\in(-c;0)$ then $f(x)<g(x)$ for all $x\in(-c;0)$.

Solution:

a.

- Assume exists $a\in(0;c)$ such that $f(a)\leq g(a)$.
- Define $h(x)=f(x)-g(x)$. Then $h(0)=0$, $h(a)\leq 0$, $h'(x)>0$ for all $x\in(0;c)$.

- Using the mean value theorem we obtain h' should be negative at some point on the interval $(0;a)$.
- Contradiction.

b.

- Assume exists $a \in (-c;0)$ such that $f(a) \geq g(a)$.
- Define $h(x) = f(x) - g(x)$. Then $h(0) = 0$, $h(a) \geq 0$, $h'(x) > 0$ for all $x \in (-c;0)$.
- Using the mean value theorem we obtain h' should be negative at some point on the interval $(0;a)$.
- Contradiction.

Problem 8.3

The sum of two numbers is 16. Find the numbers given that the sum of their cubes is an absolute minimum.

Solution:

- $x + y = 16$. Denote $f(x,y) = x^3 + y^3$
- $g(x) = f(x,16-x) = x^3 + (16-x)^3$
- $g'(x) = 3x^2 - 3(16-x)^2 = 3x^2 - 3(256 - 32x + x^2) = -753 + 96x$, critical point is $x = 8$
- $g''(x) = 96 > 0$
- So, the local minimum is an absolute minimum.

Problem 8.4

What is the maximum volume for a rectangular box (square base, no top), made from 12 square feet of cardboard?

Solution:

- Denote the base side a and the height h. So, $a^2 + 4ha = 12 \Rightarrow h = \frac{12 - a^2}{4a}$
- $V = a^2 h = a^2 \frac{12 - a^2}{4a} = 3a - \frac{1}{4}a^3$
- $\left(3a - \frac{1}{4}a^3\right)' = 3 - \frac{3}{4}a^2 = 0 \Rightarrow a = \pm 2 \Rightarrow a = 2$, $h = 1 \Rightarrow V = a^2 h = 4$.

Problem 8.5

Prove that a polynomial of degree n can have at most $n-2$ point of inflection.

Solution:

- Denote: $p(x) = \sum_{i=0}^{n} a_i x^i$

- Polynomial function has a second derivative at each point.
- That is why $p''(x)=\sum_{i=0}^{n-2}(i+2)(i+1)a_{i+2}x^i=0$ for each inflection point.
- $p''(x)=\sum_{i=0}^{n-2}(i+2)(i+1)a_i x^i=0$ - polynomial of order $n-2$ has at most $n-2$ roots.
- $p(x)=\sum_{i=0}^{n}a_i x^i$ has at most $n-2$ point of inflection.

Problem 8.6

Sketch the following graph $f(x)=\sqrt[3]{(x^2+1)^2}$

Solution:

Domain: any value of x.

Vertical and horizontal asymptotes: not found.

Intercepts:

x axis: $\sqrt[3]{(x^2+1)^2}=0$ - no solutions.

y axis: $f(0)=\sqrt[3]{(0^2+1)^2}=1$

Symmetry: $f(-x)=f(x)$

First derivative: $f'(x)=\dfrac{2x}{(x^2+1)^{\frac{1}{3}}}$

Critical points: $f'(x)=\dfrac{2x}{(x^2+1)^{\frac{1}{3}}}=0 \Leftrightarrow x=0$

Intervals of increase/decrease:

$f'(x)=\dfrac{2x}{(x^2+1)^{\frac{1}{3}}}>0 \Leftrightarrow x>0$, $\quad f'(x)=\dfrac{2x}{(x^2+1)^{\frac{1}{3}}}<0 \Leftrightarrow x<0$

Second derivative:

$$f'(x)=\left(\frac{2x}{(x^2+1)^{\frac{1}{3}}}\right)'=\frac{2(x^2+1)^{\frac{1}{3}}-2x\frac{1}{3}2x(x^2+1)^{-\frac{2}{3}}}{(x^2+1)^{\frac{2}{3}}}=\frac{2(x^2+1)^{\frac{1}{3}}-\frac{4}{3}x^2(x^2+1)^{-\frac{2}{3}}}{(x^2+1)^{\frac{2}{3}}} \Leftrightarrow f''(0)=2$$

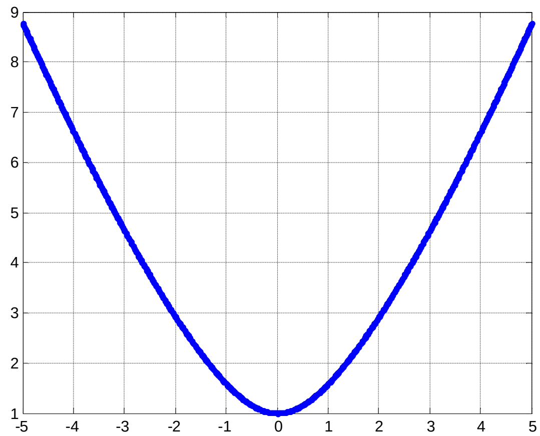
Intervals of concavity:

$$\frac{2\left(x^2+1\right)^{\frac{1}{3}}-\frac{4}{3}x^2\left(x^2+1\right)^{-\frac{2}{3}}}{\left(x^2+1\right)^{\frac{2}{3}}}>0 \Leftrightarrow \left(x^2+1\right)-\frac{2}{3}x^2>0 \text{ - always true!}$$

The function is always concave up.

Points of inflection:

$$\frac{2\left(x^2+1\right)^{\frac{1}{3}}-\frac{4}{3}x^2\left(x^2+1\right)^{-\frac{2}{3}}}{\left(x^2+1\right)^{\frac{2}{3}}}=0 \Leftrightarrow \left(x^2+1\right)-\frac{2}{3}x^2=0 \text{ - no points of inflection!}$$



Problem 8.7

Sketch the following graph $f\left(x\right)=\frac{x^2-2x}{3x^2+x}$

Solution:

- $f\left(x\right)=\frac{x^2-2x}{3x^2+x}=\frac{x-2}{3x+1}=\frac{x+\frac{1}{3}-2\frac{1}{3}}{3x+1}=\frac{1}{3}-\frac{2\frac{1}{3}}{3x+1}, \ x\neq 0$
- We can sketch the graph using the following stages:

1. $f\left(x\right)=\frac{1}{x}$ - hyperbolic function

2. $f\left(x\right)=\frac{1}{x+1}$ - move one unit left

3. $f\left(x\right)=\frac{1}{3x+1}$ - shrink in 3 times on y axis

$f(x)=\dfrac{2\frac{1}{3}}{3x+1}$ - stretch $2\frac{1}{3}$ times on x axis

4. $f(x)=-\dfrac{2\frac{1}{3}}{3x+1}$ - flip on x axis

5. $f(x)=\dfrac{1}{3}-\dfrac{2\frac{1}{3}}{3x+1}$ - move $\dfrac{1}{3}$ units up