Tutorial 9

Problem 9.1

A rectangular warehouse will have 5000 square feet of floor space and will be separated into two rectangular rooms by an interior wall. The cost of the exterior walls is $150 per linear foot and the cost of the interior wall is $100 per linear foot. Find the dimension that will minimize the cost of building the warehouse.

Solution:

- x is the length, y is the width.
- $S = x \cdot y = 5000$, $f(x, y) = 150(2x + 2y) + 100y = 300x + 400y$
- $f\left(x, \frac{5000}{x}\right) = 300x + 400 \cdot \frac{5000}{x}$
- $\left(300x + 400 \cdot \frac{5000}{x}\right)' = 300 - \frac{200000}{x^2} = 0 \Rightarrow x = \sqrt{\frac{2000}{3}}$
- $\left(300x + 400 \cdot \frac{5000}{x}\right)'' = 300 + \frac{400000}{x^3} > 0$ - local minimum.
- For $x = 0; +\infty$ we get an infinite price.
- So, $x = \sqrt{\frac{2000}{3}}$, $y = \frac{5000}{\sqrt{\frac{2000}{3}}}$ is the optimal size.

Problem 9.2

Fix a positive number P. Let R denote the set of all rectangles with perimeter P. Prove that there is a member of R that has a maximum area. What are the dimensions of the rectangle of maximum area?

Solution:

- $2x + 2y = P$
- $S = xy = x\left(\frac{P}{2} - x\right)$
- $S' = \frac{P}{2} - 2x = 0 \Rightarrow x = y = \frac{P}{4}$

Problem 9.3

Two functions are everywhere defined. Can they both be everywhere continuous:

a. if they differ only at a finite number points?

b. if they differ only on a bounded closed interval $[a;b]$?

c. if they differ only on a bounded open interval $(a;b)$?

Solution:

- Denote the functions to be f and g.

a.

- No.
- Assume $f(a) \neq g(a)$. Denote $\varepsilon = |f(a) - g(a)|$
- Then exists $\delta > 0$ such that for $x \in (a-\delta; a+\delta)$ $|f(x) - f(a)| < \frac{\varepsilon}{2}, |g(x) - g(a)| < \frac{\varepsilon}{2}$
- $f(a) - \frac{\varepsilon}{2} < f(x) < f(a) + \frac{\varepsilon}{2},\ g(a) - \frac{\varepsilon}{2} < g(x) < g(a) + \frac{\varepsilon}{2}$
- So, $f(a) - \frac{\varepsilon}{2} - \left(g(a) + \frac{\varepsilon}{2}\right) < f(x) - g(x) < f(a) + \frac{\varepsilon}{2} - \left(g(a) - \frac{\varepsilon}{2}\right)$
- $\left(f(a) - g(a)\right) - \varepsilon < f(x) - g(x) < \left(f(a) - g(a)\right) + \varepsilon$
- $\left|\left(f(x) - g(x)\right) - \left(f(a) - g(a)\right)\right| < \varepsilon \iff |f(x) - g(x)| > 0$ for all $x \in (a-\delta; a+\delta)$.
- The functions differ on an infinite number points. Contradiction.

b.

- $f - g$ is also continuous.
- So, $f(x) - g(x) = 0$ on the interval $(a; +\infty)$ then $f(a) - g(a) = 0$.
- Contradiction.

c.

- Yes.
- $f(x) = \begin{cases} 5 - x^2, & -\sqrt{5} \le x \le \sqrt{5} \\ 0, & else \end{cases}$ and $g(x) = 0$

Problem 9.4

Find an equation for all tangents to the curve $y = x^3$ that pass through the point $(0; -1)$.

Solution:

- $y' = 2x$. The tangent at the point $(x_0; x_0^3)$ is $y - x_0^3 = 2x_0(x - x_0)$
- If it passes through $(0; -1)$ then $-1 - x_0^3 = 2x_0(0 - x_0) \iff -1 - x_0^3 = -2x_0^2 \iff x_0^3 - 2x_0^2 + 1 = 0 \iff$

  $x_0 = 1$ or $x_0^2 - x_0 - 1 = 0 \iff x_0 = \frac{1 \pm \sqrt{5}}{2}$
- So, we got 3 tangents, passing through the point $(0; -1)$.

Problem 9.5

Show that $\frac{d}{dx}\left(x^{-n}\right) = -\frac{n}{x^{n+1}}$ for all possible integers n by showing that $\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x+h)^n} - \frac{1}{x^n}\right] = -\frac{n}{x^{n+1}}$

Solution:

$$\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x+h)^n} - \frac{1}{x^n}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{x^n - (x+h)^n}{(x+h)^n x^n}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{-\sum_{i=1}^{n}\binom{n}{i}x^{n-i}h^i}{(x+h)^n x^n}\right] = \lim_{h\to 0}\frac{-\sum_{i=1}^{n}\binom{n}{i}x^{n-i}h^{i-1}}{(x+h)^n x^n} = -nx^{-n-1}$$

Problem 9.6

Find the vertical and horizontal asymptotes:

a. $f(x) = \frac{x^3}{x+2}$ b. $f(x) = \frac{2x}{\sqrt{x^2-1}}$

Solution:

a.

- $\lim_{x\to\infty}\frac{x^3}{x+2} = \infty \Rightarrow$ there is no horizontal asymptote.
- $\lim_{x\to -2}\frac{x^3}{x+2} = \infty \Rightarrow x = -2$ is a vertical asymptote.

b.

- $\lim_{x\to\infty}\frac{2x}{\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{2}{\sqrt{1-\frac{1}{x^2}}} = 2 \Rightarrow y = 2$ is a vertical asymptote.
- $\lim_{x\to 1}\frac{2x}{\sqrt{x^2-1}} = \infty \Rightarrow x = 1$ is a vertical asymptote.
- $\lim_{x\to -1}\frac{2x}{\sqrt{x^2-1}} = \infty \Rightarrow x = -1$ is a vertical asymptote.