On Random Walks in Direction-Aware Network Problems

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ABSTRACT

Graph theory provides a powerful set of metrics and conceptual ideas to model and investigate the behavior of communication networks. Most graph-theoretical frameworks in the networking literature are based on undirected graph models, where a symmetric link weight is assigned to each link of the network. However, many communication networks must account for directionality of communication links. This paper reports on an effort to extend some of the existing results of symmetric graphs to asymmetric ones. In particular we are interested in the behavior of random-walk based algorithms in directed graphs and we find the average travel time of a random-walk as a function of an asymmetric Laplacian matrix, which is in turn a function of link weights.

1. INTRODUCTION

Consider a wireless network which consists of a set of nodes with known transmit powers. The Shannon capacity of each wireless link depends on the value of node transmit power, node distances, and the interference among different nodes. Depending on the arrangement of the nodes, the capacity of a link (i, j) may be different from the capacity of link (j, i), therefore, we cannot model the behavior of such networks with undirected graphs.

This paper tries to shed light on such asymmetric networks. We are particularly concerned with the behavior of random-walk models on directed graphs. Generally speaking, the theory of random walks is not limited to undirected graphs, yet most of the existing literature on properties of random walks have concentrated on reversible Markov chains which are inherently associated with an undirected symmetric graph model. In a random walk, we are usually interested in finding the probability of hitting a specific subset of graph nodes for the first time. The probability that a random walker first reaches a destination point exactly equals the solution to the Dirichlet problem with boundary conditions at the locations of the destination points on an undirected symmetric graph [1]. It is also shown that one can construct a purely resistive electrical network to solve the Dirichlet problem with boundary conditions [1]. As a result, many definitions in random walk theory and electric circuits are closely related.

Our interest in this paper is the average travel time between arbitrary pairs of nodes s and d averaged over all possible source-destination pairs. Suppose for each link $l = (i, j)$ there is a cost $z_l = z(i, j)$. After a random-walk starts from source node s, at each step it traverses one link, incurs a cost, and continues until it is absorbed at destination d. It has been already shown that in an undirected graph with symmetric link weights the average cost of this journey between any two random nodes is equal to $\frac{1}{2}\hat{\tau}\sum_k(\sum_j w_{kj} z(k, j))$, where $\hat{\tau}$ is the average network criticality or average resistance distance of the graph and can be written in terms of the components of the undirected Moore-Penrose Laplacian matrix: $\hat{\tau} = \frac{2}{n-1}Tr(L^+)$ [2]. In the present paper we develop a mathematical framework to calculate the average travel cost of a random-walk in a directed graph using a directed Laplacian matrix. Our focus in this paper is on the case of unit link costs ($z(i, j) = 1\ \forall(i, j) \in E$). We will discuss the general case in subsequent research reports.

2. MAIN RESULTS

We start by introducing the graph model. A directed graph $G(V, E, W)$ consists of a finite node set V which contains n nodes, a link set $E \in V \times V$, and a set of positive link weights w_{ij} (matrix W). The out-weight of a node i is defined as $W_i^o = \sum_{j,i->j} w_{ij}$. We can define a transition probability matrix $P = [p_{ij}]$ of an irrecucible Markov random walk through the directed graph which satisfies $\sum_{j,i->j} p_{ij} = 1\ \forall i \in V$. We also assume the stationary distribution for each node i is π_i, where $\sum_i \pi_i = 1$ ($\pi_i > 0$ for connected graphs). In this paper we consider a connected graph and use the natural (weighted) random walk whose transition probability is defined as $p_{ij} = \frac{w_{ij}}{W_i^o}$. We define the combinatoric Laplacian of a graph as: $L = \Phi(I - P)$ (inspired by [3]), where Φ is a diagonal matrix with main diagonal equal to the stationary probability vector of the graph (vector $\overrightarrow{\pi}$).

The average travel time equals the average of the commute time between arbitrary source-destination pairs. Therefore, we first find the commute time of a random-walk in terms of its combinatoric Laplacian as defined above. Following [4] we introduce the fundamental matrix Z of a Markov chain by $Z = (I - (P - J\Phi))^{-1}$, where J is an square matrix whose entries are all equal to 1. Let S denote the hitting time matrix of the Markov chain, in [4] it is proved that $S = (Jdiag(Z) - Z)\Phi^{-1}$. Let $X = Z\Phi^{-1}$ (in this paper $diag(Z)$ denotes a diagonal matrix whose main diagonal components are equal to the main diagonal components of Z), then considering the fact that Φ is a diagonal matrix we conclude that $S = Jdiag(X) - X$, therefore, the commute time matrix is equal to $C = Jdiag(X) + diag(X)J - X - X^t$, where X^t denotes the transpose of X.

LEMMA 2.1. *The following statements are true for matrices Z, X, L, and Φ.*

$$
\begin{aligned}
J\Phi J &= J && (1)\\
ZJ &= J && (2)\\
LJ &= JL = 0 && (3)\\
X\Phi J &= J\Phi X = J && (4)\\
LXL &= L, \quad XLX = X - J && (5)\\
L &= X^{-1} - \Phi J\Phi && (6)
\end{aligned}
$$

PROOF. All the above equations can be simply derived. Here we show the proof of equations (3) and (6). For equation (3) we have:

$$LJ = \Phi(I-P)J = \Phi(J - PJ) = \Phi(J - J) = 0$$

Let $\overrightarrow{1} = (1, 1, ..., 1)^t$. To prove $JL = 0$, we consider one row of the matrix product JL:

$$\overrightarrow{1}^t L = \overrightarrow{1}^t \Phi(I-P) = \overrightarrow{\pi}^t (I-P) = \overrightarrow{\pi}^t - \overrightarrow{\pi}^t = 0$$

For the last equation we can write:

$$
\begin{aligned}
X^{-1} &= (Z\Phi^{-1})^{-1} = \Phi(I - P + J\Phi)\\
&= \Phi - \Phi P + \Phi J\Phi = L + \Phi J\Phi
\end{aligned}
$$

□

Equation (5) suggests that the Moore-Penrose inverse of Laplacian is related to X. Lemma 2.2 explores this relationship.

LEMMA 2.2. *X and L^+ are related according to the following equations:*

$$
\begin{aligned}
X &= (I - J\Phi)L^+(I - \Phi J) + J && (7)\\
L^+ &= (I - \frac{J}{n})X(I - \frac{J}{n}) && (8)
\end{aligned}
$$

PROOF. We use the following fact to prove equation (7).

FACT 2.3. *Suppose $rank(A+B) = rank(A) + rank(B)$, then:*

$$(A+B)^+ = (I - C^+B)A^+(I - BC^+) + C^+$$

where $C = (I - AA^+)B(I - A^+A)$ (See [5]).

Let $A = L$ and $B = \Phi J\Phi$. We use Fact 2.3 and equation (6) to find an expression for X based on L^+. Since $A = L$, it is easy to see that $I - AA^+ = I - A^+A = \frac{1}{n}J$. Using equation (1) we conclude $C = \frac{1}{n^2}J$ and $C^+ = J$. Therefore

$$
\begin{aligned}
(L + \Phi J\Phi)^+ &= (I - J\Phi J\Phi)L^+(I - \Phi J\Phi J) + J\\
X &= (I - J\Phi)L^+(I - \Phi J) + J
\end{aligned}
$$

where we have used equation (1) and the fact that $L + \Phi J\Phi$ is invertible. We need the following fact to prove equation (8).

FACT 2.4. *Suppose matrix A is invertible and $A + B$ is singular, then:*

$$(A+B)^+ = (I + A^{-1}B)^+(A^{-1} + A^{-1}BA^{-1})(I + BA^{-1})^+$$

See [5] for proof.

Let $A = X^{-1}$ and $B = -\Phi J\Phi$, then we can use Fact 2.4 to expand equation (6):

$$
\begin{aligned}
L^+ &= (I - X\Phi J\Phi)^+(X - X\Phi J\Phi X)(I - \Phi J\Phi X)^+\\
&= (I - J\Phi)^+(X - J)(I - \Phi J)^+ && (9)
\end{aligned}
$$

We have used equation (4) to obtain (9). In order to find the Moore-Penrose inverse of $I - J\Phi$ and $I - \Phi J$ we use the following fact.

FACT 2.5. *Suppose matrix A is Hermitian and nonsingular, and Let x and y be column vectors.*

$$(A + xy^*)^+ = (I - aa^+)A^{-1}(I - bb^+)$$

where $a = A^{-1}x$ and $b = A^{-1}y$, and y^* denotes conjugate transpose of vector y [5].

Now, we note that:

$$I - J\Phi = I - \overrightarrow{1}\overrightarrow{\pi}^t = -(-I + \overrightarrow{1}\overrightarrow{\pi}^t) \quad (10)$$

Let $A = -I$, $x = \overrightarrow{1}$, and $y = \overrightarrow{\pi}$. Now we can use Fact 2.5 to simplify equation (10).

$$
\begin{aligned}
a &= -I\overrightarrow{1} = -\overrightarrow{1}, \quad a^+ = -\frac{1}{n}\overrightarrow{1}^t\\
b &= -I\overrightarrow{\pi} = -\overrightarrow{\pi}, \quad b^+ = -\frac{1}{||\overrightarrow{\pi}||^2}\overrightarrow{\pi}^t
\end{aligned}
$$

Therefore

$$(I - J\Phi)^+ = (I - \frac{J}{n})(I - \frac{1}{||\overrightarrow{\pi}||^2}\overrightarrow{\pi}\overrightarrow{\pi}^t) \quad (11)$$

Similarly

$$(I - \Phi J)^+ = (I - \frac{1}{||\overrightarrow{\pi}||^2}\overrightarrow{\pi}\overrightarrow{\pi}^t)(I - \frac{J}{n}) \quad (12)$$

Now we can employ equations (11), (12) in (9) to simplify L^+.

$$
\begin{aligned}
L^+ &= (I - \frac{J}{n})(I - \frac{1}{||\overrightarrow{\pi}||^2}\overrightarrow{\pi}\overrightarrow{\pi}^t)(X - J) \times\\
&\quad (I - \frac{1}{||\overrightarrow{\pi}||^2}\overrightarrow{\pi}\overrightarrow{\pi}^t)(I - \frac{J}{n}) && (13)
\end{aligned}
$$

On the other hand, using equation (4) and (1) and considering the fact that $\overrightarrow{\pi}\overrightarrow{\pi}^t = \Phi J\Phi$ one can see:

$$
\begin{aligned}
(I - \frac{1}{||\overrightarrow{\pi}||^2}\overrightarrow{\pi}\overrightarrow{\pi}^t)(X - J) &= X - J - \frac{1}{||\overrightarrow{\pi}||^2}\Phi J\Phi X\\
&\quad + \frac{1}{||\overrightarrow{\pi}||^2}\Phi J\Phi J\\
&= X - J && (14)
\end{aligned}
$$

Similarly

$$(X-J)(I-\frac{1}{||\overrightarrow{\pi}||^2}\overrightarrow{\pi}\,\overrightarrow{\pi}^t)=X-J \qquad (15)$$

Finally, it is easy to verify that:

$$(I-\frac{J}{n})(X-J)=(I-\frac{J}{n})X \qquad (16)$$

Combining equations (13), (14), (15), (16) we get (8). □

LEMMA 2.6. *Commute Matrix C is equal to: $C=[c_{ij}]=Jdiag(L^+)+diag(L^+)J-L^+-(L^+)^t$, or equivalently $c_{ij}=l^+_{ii}+l^+_{jj}-l^+_{ij}-l^+_{ji}$.*

PROOF. Equation (7) can be written as:

$$X \;=\; L^+ + (1+\alpha)J - J\Phi L^+ - L^+\Phi J \qquad (17)$$

where $\alpha=\overrightarrow{\pi}^t L^+ \overrightarrow{\pi}$.

$$vec(diag(X)) = vec(diag(L^+)) + (1+\alpha)\overrightarrow{1} - \\ -(L^+)^t\overrightarrow{\pi} - L^+\overrightarrow{\pi} \qquad (18)$$

where $vec(diag(X))$ denotes a column vector whose entries are equal to the diagonal entries of $diag(X)$. We multiply two sides of equation (18) from right by $\overrightarrow{1}^t$. Note that $vec(diag(X)) * \overrightarrow{1}^t = diag(X)J$, $vec(diag(L^+)) * \overrightarrow{1}^t = diag(L^+)J$, and $\overrightarrow{\pi} * \overrightarrow{1}^t = \Phi J$, thus:

$$diag(X)J = diag(L^+)J + (1+\alpha)J - (L^+)^t\Phi J - L^+\Phi J$$

Similarly

$$Jdiag(X) = Jdiag(L^+) + (1+\alpha)J - J\Phi(L^+)^t - J\Phi L^+$$

It is enough to use equation $C = Jdiag(X) + diag(X)J - X - X^t$. By replacing the values of $Jdiag(X)$, $diag(X)J$, X, and X^t in terms of L^+ one can easily conclude $C = Jdiag(L^+) + diag(L^+)J - L^+ - (L^+)^t$. □

Now we are ready to state the main result.

THEOREM 2.7. *The average random-walk travel time between every arbitrary pair of nodes in a directed graph is equal to $\frac{2}{n-1}Tr(L^+)$, where $L=\Phi(I-P)$ is the combinatoric Laplacian of the directed graph.*

PROOF. First we note that $L^+J = JL^+ = 0$. This can be easily seen using the fact that $(I-\frac{J}{n})J = J(I-\frac{J}{n})$ (note that $JJ=nJ$) and considering equation (8). Now we can find the average travel time T by averaging over commute times between all possible node pairs.

$$\begin{aligned} T &= \frac{1}{n(n-1)}\sum_{(i,j}c_{ij} = \frac{1}{n(n-1)}\overrightarrow{1}^t C\overrightarrow{1} \\ &= \frac{1}{n(n-1)}(\overrightarrow{1}^t(Jdiag(L^+)\overrightarrow{1} + \overrightarrow{1}^t diag(L^+)J\overrightarrow{1} - \\ &- \overrightarrow{1}^t L^+\overrightarrow{1} - \overrightarrow{1}^t(L^+)^t\overrightarrow{1}) \\ &= \frac{1}{n(n-1)}(nTr(L^+) + nTr(L^+) - 0 - 0) \\ &= \frac{2}{n-1}Tr(L^+) \qquad (19) \end{aligned}$$

□

Equation (19) has the same form of average network criticality (or resistance distance) [2], but here we use the combinatoric asymmetric Laplacian.

3. CONCLUSIONS AND ROAD MAP

We extended the existing mathematics to find attributes of random-walk based algorithms in undirected graphs for direction-aware asymmetric networks. We were specially interested in the average travel time of a random-walk in a directed graph and developed a machinery to find this quantity using the trace of an asymmetric directed Laplacian.

Theorem 2.7 provides us with a clear relationship between combinatoric Laplacian of directed graphs and the average travel time of a random-walk. While equation (19) is similar to the equation for average resistance distance in undirected graphs, the nature of the Laplacian in the directed case is different. For a connected graph in undirected case the Laplacian (and its generalized inverse) is a positive semi-definite matrix and the trace of generalized inverse Laplacian is a strictly convex and decreasing function of link weights. Therefore, one can easily use a convex optimization problem to find optimized link weights.

We plan to investigate similar properties of the asymmetric combinatorial (and weighted) Laplacian. While L is not necessarily positive semi-definite, but $L_1 = \frac{L^+ + (L^+)^t}{2}$ is positive semi-definite (since L_1 is symmetric with non-negative real eigenvalues [6]), therefore, $Tr(L_1) = Tr(L^+)$ is a convex function of L_1 [5], but its convexity with respect to link weights is not necessarily preserved because the entries of L_1 are fractional polynomial functions of link weights (note that $\Phi(i,i) = \frac{M_i}{\sum_k M_k}$ where M_i is the i^{th} principal minor determinant of matrix $I-P$ [7]). We intend to study convexity properties of $Tr(L_1)$ and to develop appropriate optimization problems and algorithms to minimize average travel time in wireless interference-aware networks.

4. REFERENCES

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