A Theory of Stationary Trees and the Balanced Baumgartner-Hajnal-Todorcevic Theorem for Trees

by

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Abstract

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Building on early work by Stevo Todorcevic, we develop a theory of stationary subtrees of trees of successor-cardinal height. We define the diagonal union of subsets of a tree, as well as normal ideals on a tree, and we characterize arbitrary subsets of a non-special tree as being either stationary or non-stationary.

We then use this theory to prove the following partition relation for trees:

Main Theorem. *Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then*

$$\text{non-}\big(2^{<\kappa}\big)\text{-special tree } \to (\kappa + \xi)^2_k .$$

This is a generalization to trees of the Balanced Baumgartner-Hajnal-Todorcevic Theorem, which we recover by applying the above to the cardinal $(2^{<\kappa})^+$, the simplest example of a non-$(2^{<\kappa})$-special tree.

As a corollary, we obtain a general result for partially ordered sets:

Theorem. *Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Let P be a partially ordered set such that $P \to (2^{<\kappa})^1_{2^{<\kappa}}$. Then*

$$P \to (\kappa + \xi)^2_k .$$

Dedication

This thesis is dedicated in memory of two special mathematicians:

Professor Paul Charles Rosenbloom, March 31, 1920 – May 26, 2005

Daniel Tsur Pasher, August 22, 1978 – July 16, 2005

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Finally, I express thanks and appreciation to God, for giving me the ability and opportunity to emulate His creative ways by producing this contribution to the body of mathematical knowledge.[1]

[1] As Rabbi Joseph B. Soloveitchik writes in *The Lonely Man of Faith* [47, pp. 17–18]:

> He engages in creative work, trying to imitate his Maker (*imitatio Dei*). The most characteristic representative of Adam the first is the mathematical scientist who whisks us away from the array of tangible things, from color and sound, from heat, touch, and smell which are the only phenomena accessible to our senses, into a formal relational world of thought constructs, the product of his "arbitrary" postulating and spontaneous positing and deducing. This world, woven out of human thought processes, functions with amazing precision and runs parallel to the workings of the real multifarious world of our senses. The modern scientist does not try to explain nature. He only duplicates it. In his full resplendent glory as a creative agent of God, he constructs his own world and in mysterious fashion succeeds in controlling his environment through manipulating his own mathematical constructs and creations.

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Chapter 1

Introduction and Background

1.1 Partition Calculus

Partition calculus, as a discipline within set theory, was developed by Erdős and Rado in their seminal paper appearing more than fifty years ago [13], a paper that appears in the short list *Classic Papers in Combinatorics* (see [17]). Because of its fundamentally combinatorial character, the Erdős-Rado partition calculus has become a highly active area of research with many connections to other fields of mathematics. The original motivation of the Erdős-Rado partition calculus was the attempt to extend the famous Ramsey Theorem (Theorem 2 below) from the first infinite ordinal ω to other ordinals, with particular emphasis on obtaining the optimal results. This preoccupation of getting the optimal results is in turn motivated by the famous and still largely unsolved problem of computing "Ramsey numbers" in the case of the finite Ramsey Theorem (Theorem 1 below).

The infinite case offers a rich theory with many surprising and deep results, surveyed in texts such as [25, chapter 15], [54], [12], [20], and [35], while the finite case is surveyed extensively in [18]. However, the primary focus of the early development of partition calculus was exclusively on linear (total) order types, including cardinals and ordinals as specific examples. It wasn't until the 1980s that Todorcevic [52] pioneered the systematic study of partition relations for partially ordered sets, although the extension of the partition calculus to non-linear order types began with Galvin [16] and the idea was anticipated even by Erdős and Rado [13, p. 430].

It is the continuation of this study which forms the main focus of this thesis. That is, we are looking to verify the truth or falsity of Ramsey-type partition relations for a partial order P.

As we shall see (section 4.5), Todorcevic showed that partition relations for partially ordered sets in general can be reduced to the corresponding partition relations for trees. Furthermore, as Todorcevic writes in [52, p. 13],

> It turns out that partition relations for trees are very natural generalizations of partition relations for cardinals and that several well-known partition relations for cardinals are straightforward consequences of the corresponding relations for trees.

This motivates our continuing of Todorcevic's study of the partition calculus for trees.

1.2 Partition Relations for Cardinals

In this section we describe some of the main results in partition calculus, in order to set the stage for our main contribution to the field.

The famous Ramsey Theorem has two forms, the finite and the infinite:

Theorem 1 (Finite Ramsey Theorem [41, Theorem B]). *For all positive integers m, r, and k, there is some positive integer n such that*

$$n \to (m)^r_k .$$

Theorem 2 (Infinite Ramsey Theorem [41, Theorem A]). *For all positive integers r and k, we have*

$$\omega \to (\omega)^r_k .$$

Both of the above theorems can be considered as generalizations of appropriate versions of the pigeonhole principle, which we recover from them by setting $r = 1$.

Moving to the uncountable cardinals, we recall that Erdős and Rado proved the following Stepping-Up Lemma:

Lemma 3 (Stepping-Up Lemma [12, Lemma 16.1], [20, Theorem 2.7]). *Let r be any positive integer, let κ be any infinite cardinal, let μ be any cardinal, and for each $\gamma < \mu$ let α_γ be any ordinal. If*

$$\kappa \to (\alpha_\gamma)^r_{\gamma<\mu}$$

then we have

$$\left(2^{<\kappa}\right)^+ \to (\alpha_\gamma + 1)^{r+1}_{\gamma<\mu} .$$

Applying the Stepping-Up Lemma to the infinite Ramsey Theorem, we obtain:

Theorem 4 ([13, p. 472]). *For all positive integers r and k, we have*

$$\omega_1 \to (\omega + 1)^r_k .$$

The following infinite version of the pigeonhole principle is essentially a restatement of the definition of cofinality:

Theorem 5. *Let κ be any infinite cardinal. For any cardinal $\mu < \mathrm{cf}(\kappa)$, we have*

$$\kappa \to (\kappa)^1_\mu .$$

In order to state the general version of the Erdős-Rado theorem, we recall the notation for *iterated exponentiation*, introduced in [12, p. 18]:

Definition 1. *For any infinite cardinal κ, we define the expression $\exp_n(\lambda)$ by recursion over n, by setting*

$$\begin{aligned} \exp_0(\lambda) &= \lambda; \\ \exp_{n+1}(\lambda) &= \exp_n\left(2^\lambda\right) = 2^{\exp_n(\lambda)}, && \text{for } n \geq 0. \end{aligned}$$

The following general version of the Balanced Erdős-Rado Theorem is obtained from Theorem 5 by recursively applying the Stepping-Up Lemma 3:

Theorem 6 ([20, Theorem 2.9])**.** *Let $r \geq 2$ be an integer, and let κ be any infinite cardinal. For any cardinal $\mu < \mathrm{cf}(\kappa)$, we have*

$$\left(\exp_{r-2}\left(2^{<\kappa}\right)\right)^{+} \to (\kappa + (r-1))^{r}_{\mu}.$$

Various weaker versions of the Balanced Erdős-Rado Theorem are more commonly stated in the literature. In particular, the following version, where $\kappa = \mu^{+}$ is a successor cardinal, appears as [25, Exercise 15.28] and [20, Corollary 2.10]:

Corollary 7. *Let r be any positive integer. For any infinite cardinal μ, we have*

$$\left(\exp_{r-1}(\mu)\right)^{+} \to \left(\mu^{+} + (r-1)\right)^{r}_{\mu}.$$

Proof. When $r = 1$ this is just Theorem 5, or in other words, the fact that the successor cardinal μ^{+} is regular.

For $r \geq 2$ this is just Theorem 6 applied to μ^{+}. □

Furthermore, sources dealing primarily with partition relations for *cardinals* rather than ordinals often omit the ordinal addition of $(r-1)$ from the goal of Corollary 7, such as [21, Exercise 9.2], [25, Theorem 15.13], [29, Exercise III.8.13], and [54, Theorem 2.3.3].

The Balanced Erdős-Rado Theorem (Theorem 6) is sharp, in the sense that the cardinal resource $(\exp_{r-2}(2^{<\kappa}))^{+}$ cannot, *in general*, be replaced by any smaller ordinal (see, for example, Corollary 56 later). However, there is much to say about extending the ordinal goals of Theorem 6, especially in the case $r = 2$. Baumgartner, Hajnal, and Todorcevic have shown that, provided κ is regular, the ordinal goal can be increased, at the cost of requiring that the number of colours be finite:

Theorem 8 (Balanced Baumgartner-Hajnal-Todorcevic Theorem [4, Theorem 3.1])**.** *Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then*

$$\left(2^{<\kappa}\right)^{+} \to (\kappa + \xi)^{2}_{k}.$$

We shall return to Theorem 8 later in chapter 4, as the generalization of this theorem to trees will form a central topic of our thesis.

What about unbalanced partition relations, where the ordinal goals are not all identical?

The earliest result was proved by Erdős, Dushnik, and Miller:

Theorem 9 (Erdős-Dushnik-Miller Theorem [11, Theorem 5.22, p. 606])**.** *For any infinite cardinal κ, we have*

$$\kappa \to (\kappa, \omega)^{2}.$$

The ordinal goal ω was extended to $\omega + 1$ by Erdős and Rado, first for successor cardinals in [13, Corollary 1 of Theorem 34, p. 459], and then for all uncountable *regular* cardinals in [12, Theorem 11.3]:

Theorem 10. *For any uncountable regular cardinal κ, we have*

$$\kappa \to (\kappa, \omega + 1)^{2}.$$

We also have the following unbalanced version of the Erdős-Rado Theorem for pairs:

Theorem 11 ([20, Theorem 3.10])**.** *Let κ be any infinite cardinal. For any cardinal $\mu < \mathrm{cf}(\kappa)$, we have*

$$\left(2^{<\kappa}\right)^+ \to \left(\left(2^{<\kappa}\right)^+, (\mathrm{cf}(\kappa)+1)_\mu\right)^2 .$$

What about extending the ordinal goal of Theorem 10 beyond $\omega+1$? The following contrasting results show that this question is sensitive to the axiomatic framework:

Theorem 12.

$$\begin{aligned} CH &\implies \omega_1 \not\to (\omega_1, \omega+2)^2 && \text{[19, Theorem 5]} \\ PFA &\implies \omega_1 \to (\omega_1, \alpha)^2 \text{ for all } \alpha < \omega_1 && \text{[50]} \end{aligned}$$

We now focus on partitioning finite subsets of ω_1, with the goal of obtaining countable homogeneous sets.

For pairs, Baumgartner and Hajnal extended the ordinal goals as far as possible:

Theorem 13 (Baumgartner-Hajnal Theorem [3])**.** *For all $\alpha < \omega_1$ and $k < \omega$, we have*

$$\omega_1 \to (\alpha)^2_k .$$

In dimensions 4 and higher, the problem is also completely resolved: The positive direction is given by Theorem 4 mentioned above, while in the negative direction Arthur Kruse has shown:[1]

Theorem 14 ([28, Theorem 19])**.** *For any $r \geq 4$, we have*

$$\omega_1 \not\to (\omega+2, r+1)^r .$$

The problem of achieving similar results in dimension 3 is however still largely open, in spite of the fifty-year-long research efforts. Theorem 4 applies here, namely we have, for any positive integer k,

$$\omega_1 \to (\omega+1)^3_k ,$$

but in the case of triples our only known limitation is the negative result

$$\omega_1 \not\to (\omega+2, \omega)^3 ,$$

which we shall prove later (Theorem 34). The lack of sufficient results in dimension 3 has led many to investigate the Erdős-Rado Conjecture, which states that

$$(\forall \alpha < \omega_1)\, (\forall n < \omega) \left[\omega_1 \to (\alpha, n)^3\right] .$$

Eric C. Milner and Karel Prikry were able to prove [38]

$$(\forall m < \omega) \left[\omega_1 \to (\omega+m, 4)^3\right] .$$

[1] Jones [23, page 1198, (6)(d)] as well as Milner-Prikry [38, page 488, (1.2)] mistakenly attribute this result to [13].

Then, working in the 1980s using a result of Todorcevic, they improved this result slightly to:

Theorem 15 ([39]).

$$\omega_1 \to (\omega + \omega + 1, 4)^3 .$$

Subsequently, Albin Jones proved:

Theorem 16 ([23]). *For all $m, n < \omega$,*

$$\omega_1 \to (\omega + m, n)^3 .$$

These are the strongest currently known results in the direction of the Erdős-Rado Conjecture described above.

1.3 Background of Special Aronszajn Trees: Historical, but Somewhat Incomplete

As we mentioned in section 1.1, trees play a central role in the study of partition relations for partially ordered sets. We shall see in the next section that it is *non-special* trees in particular that will be most important. To understand the history of non-special trees, we begin by discussing some background on special trees: What's so *special* about them?

The systematic study of set-theoretic trees was pioneered by Đuro Kurepa in the 1930s [30], in the context of examining Souslin's Problem.[2] Souslin's Problem goes back to 1920 [48], and its most succinct formulation is:

> Is every linearly ordered topological space satisfying the countable chain condition (ccc) necessarily separable?

A counterexample would be called a *Souslin line*, while the conjecture that the answer is "yes" (meaning that a Souslin line does *not* exist) is called *Souslin's Hypothesis* (SH).

In the course of attempting to prove SH, Kurepa showed in 1935 [30] that the problem can be reformulated in terms of trees,[3] and thus "eliminated topological considerations from Souslin's Problem and reduced it to a problem of combinatorial set theory" [27, p. 3]:

Definition 2. *A tree T is* Souslin *if:*

- *it has height ω_1,*
- *every chain is countable, and*
- *every antichain is countable.*

[2] See also Todorcevic's description of Kurepa's work on trees in [34, pp. 6–11], as well as the survey article [51] covering Kurepa's work and related material.

[3] Kurepa states the equivalence given by our Theorem 17 explicitly in [30, §12.D.2, pp. 124–125] [34, p. 111] and [31, Section 8, p. 134] [34, p. 119], with the proof given by the equivalence $P_2 \iff P_5$ of the Fundamental Theorem in the Appendix [30, §C.3, p. 132].

Surprisingly, several sources [46, Section 2.1, p. 202] [42, p. 1116] attribute the reformulation to E. W. Miller in 1943 [36]. Others ([27, p. 3], Todorcevic in [34, p. 9]) acknowledge that Miller rediscovered Kurepa's result.

Theorem 17.

$$\exists \textit{ Souslin line} \iff \exists \textit{ Souslin tree.}$$

We now know that Souslin's Problem is independent of ZFC. Among other constructions, we have:[4]

Theorem 18.

$$\begin{aligned} \diamond &\implies \exists \textit{ Souslin tree} && \textit{[22, Theorem 6.2]} \\ MA_{\aleph_1} &\implies \nexists \textit{ Souslin tree} && \textit{[46, Lemma 7.2]} \end{aligned}$$

However, if we weaken the definition slightly, we can construct Aronszajn trees (or, as Mary Ellen Rudin calls them [42, p. 1115], *fake Souslin trees*) with no special axioms:

Definition 3. *A tree T is* Aronszajn *if:*

- *it has height ω_1,*
- *every chain is countable, and*
- *every* level *is countable.*

Nachman Aronszajn (credited in [30, §9, footnote 3, p. 96] [34, footnote 21, p. 88]), Kurepa [30, §9.5, Theorem 6, p. 96] [34, p. 88], and others showed:

Theorem 19. *Aronszajn trees exist.*

When constructing an Aronszajn tree, a natural question to ask — either in the presence of special axioms, or before the independence results were known, even without any special axioms — is whether the tree is Souslin. Kurepa observed that an Aronszajn tree may fail to be Souslin for a very *special* reason, which can be formulated using either of the following equivalent conditions:

Theorem 20 ([32, Theorem 1, p. 837] [34, p. 172])**.** *For any partially ordered set $\langle P, <_P \rangle$, the following are equivalent:*[5]

1. *There is a function $f : P \to \mathbb{Q}$ such that*

$$s <_P t \implies f(s) < f(t)$$

(that is, $f : \langle P, <_P \rangle \to \langle \mathbb{Q}, < \rangle$ is an order-homomorphism)[6]*;*

2. *P is a union of countably many antichains, that is, we can write*

$$P = \bigcup_{n<\omega} A_n,$$

where each A_n is an antichain.

[4] In fact, both the Diamond Principle ($\diamond$) and Martin's Axiom (MA) were initially formulated by extracting the combinatorial content from the consistency proofs related to Souslin's Problem.

[5] Various proofs are given in [32] [34], [29, Lemma III.5.17], [51, Theorem 9.1], and [25, Lemma 14.12], though the proof in [25] needs to be corrected by adding the condition $n \leq \pi(t)$ to the subscript.

Surprisingly, Jech [21, p. 123] attributes this equivalence to Galvin, without citing any particular source. Todorcevic [51, Remark 9.15(i)] clarifies that "This theorem was discovered independently and later by Galvin (unpublished)."

[6] This is often described by saying "$\langle P, <_P \rangle$ is $\langle \mathbb{Q}, < \rangle$-embeddable", but this is an unfortunate use of the term *embeddable*, as we do not require f to be injective, so that it is not an embedding in the usual sense.

Definition 4. *An Aronszajn tree T is called a* special Aronszajn tree *if it satisfies either (and therefore both) of the equivalent conditions of Theorem 20.*

It is clear that a special Aronszajn tree cannot be Souslin, as it is impossible for an uncountable tree to be a union of countably many antichains all of which are countable.

Further strengthening Theorem 19, Kurepa showed in [31, Section 27, p. 156] [34, p. 138] that:

Theorem 21. *There exists a special Aronszajn tree.*

Baumgartner, Malitz, and Reinhardt showed that assuming $\mathrm{MA}_{\aleph_1}$, not only are there no Souslin trees, but:

Theorem 22 ([5, Theorem 4])**.**

$$MA_{\aleph_1} \implies \textit{every Aronszajn tree is special.}$$

This may give the impression that nonspecial trees are somewhat pathological. However, this is only because until now we have restricted our attention to Aronszajn trees, so that our understanding of special and nonspecial trees is somewhat incomplete.

1.4 Todorcevic's Paradigm Shift: Nonspecial Trees

In Kurepa's work on trees, motivated by the quest to resolve Souslin's Problem, the main classification of trees was by their width [30, §8.A.11, pp. 75–76] [34, pp. 71–72], with a special focus on Aronszajn trees. So although Kurepa proved Theorem 20 for all partial orders, the distinction between special and nonspecial was generally considered (by Kurepa and his successors) only for Aronszajn trees.

But being Aronszajn is mainly a condition on the width of the tree, the cardinality of its levels; being special or non-special is a distinction in the number of its antichains, in some sense related to the height of the tree. We can consider one without the other.

It was Stevo Todorcevic who pioneered the systematic study of nonspecial trees without regard to their width, in his early work in the late 1970s [49, 52]. With this paradigm shift, he was the first to properly understand the notion of nonspecial trees and put it into the right context inside the whole of set theory. We can forget about trees being Aronszajn or Souslin, and simply define what it means for trees of height ω_1 to be special or nonspecial, regardless of their width:[7]

Definition 5 ([49, p. 250])**.** *A tree T is a* special tree *if it can be written as a union of countably many antichains. (See the equivalent conditions in Theorem 20.) Otherwise, T is a* nonspecial tree.

In some sense the class of nonspecial trees represents a natural generalization of the first uncountable ordinal ω_1, which in turn can be considered the simplest example of a nonspecial tree. Todorcevic showed that many partition relations known to be true for ω_1 are true for nonspecial trees as well. And in contrast to our previous observation (Theorem 22) that nonspecial *Aronszajn* trees may not exist, Kurepa showed [33, Theorem 1] [34, p. 236] that there *does* exist a nonspecial tree with no uncountable

[7]Unfortunately, it remains common [29, Definition III.5.16] [21, p. 117] [25, p. 41] to define special Aronszajn trees only, rather than defining special and nonspecial trees more broadly as introduced by Todorcevic. A significant early exception is Fremlin's book [14, §A3I(d)], where the broader term *special tree* is defined (though all relevant applications in that text are stated in terms of special *Aronszajn* trees).

chain, namely $w\mathbb{Q}$ (and its variant $\sigma\mathbb{Q}$), the collection of all well-ordered subsets of $\mathbb{Q}$, ordered by end-extension. Thus our generalization from ω_1 to nonspecial trees is not vacuous.

We can examine a similar generalization for heights greater than ω_1:

Definition 6 ([51, p. 246], [52, p. 4, p. 15ff.])**.** *For any infinite cardinal κ, a tree T is a* κ-special tree *if it can be written as a union of $\leq \kappa$ many antichains. Otherwise, T is a* non-κ-special tree*.*

Again, the class of non-κ-special trees represents a natural generalization of the ordinal κ^+, which in turn can be considered the simplest example of a non-κ-special tree. And again, Todorcevic showed that many partition relations known to be true for an arbitrary successor cardinal κ^+ are true for non-κ-special trees as well. Of course, the *nonspecial tree* of Definition 5 is really a non-$\aleph_0$-special tree, but we omit the cardinal in that case for convenience.

In chapter 3 we shall describe a new theory of stationary subtrees of a nonspecial tree. We shall define the diagonal union of subsets of a tree, as well as normal ideals on a tree, and we characterize arbitrary subsets of a non-special tree as being either stationary or non-stationary.

Then, in chapter 4, we shall use this theory to prove a partition relation for trees, Theorem 29, which is a generalization to trees of the Balanced Baumgartner-Hajnal-Todorcevic Theorem for cardinals mentioned earlier (Theorem 8).

1.5 Partition Relations for Nonspecial Trees

Todorcevic showed in [52] that many of the partition relations known to be true for cardinals, including several of those that we mentioned in section 1.2, have natural generalizations to nonspecial trees. We summarize some of those results here. Recall that, as we shall discuss in section 4.5, partition relations for trees imply corresponding partition relations for partially ordered sets in general, making the study of trees a very powerful endeavour.

The following generalizes the Stepping-Up Lemma (Lemma 3):

Theorem 23 (Stepping-Up Lemma for Trees [52, Theorem 24])**.** *Let r be any positive integer, let κ be any infinite cardinal, let μ be any cardinal, and for each $\gamma < \mu$ let α_γ be any ordinal. If*

$$\kappa \to (\alpha_\gamma)^r_{\gamma<\mu}$$

then we have

$$\textit{non-}\big(2^{<\kappa}\big)\textit{-special tree} \to (\alpha_\gamma + 1)^{r+1}_{\gamma<\mu}\,.$$

Applying the Stepping-Up Lemma for Trees to the infinite Ramsey Theorem (Theorem 2), we obtain the following generalization of Theorem 4:

Theorem 24 ([16, Theorem 9(3)], [52, Theorem 29])**.** *For all positive integers r and k, we have*

$$\textit{nonspecial tree} \to (\omega + 1)^r_k\,.$$

Applying the Stepping-Up Lemma for Trees recursively to Theorem 5, we obtain the Balanced Erdős-Rado Theorem for Trees, generalizing Theorem 6:

Theorem 25. *Let $r \geq 2$ be an integer, and let κ be any infinite cardinal. For any cardinal $\mu < \mathrm{cf}(\kappa)$, we have*

$$\textit{non-}\left(\exp_{r-2}\left(2^{<\kappa}\right)\right)\textit{-special tree} \rightarrow \left(\kappa + (r-1)\right)^r_\mu .$$

The unbalanced version of the Erdős-Rado Theorem (Theorem 11) has the following generalization to trees:

Theorem 26 ([52, Theorem 15])**.** *Let κ be any infinite cardinal. For any cardinal $\mu < \mathrm{cf}(\kappa)$, we have*

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree} \rightarrow \left(\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree}\ , \left(\mathrm{cf}(\kappa)+1\right)_\mu\right)^2 .$$

In particular, applying Theorem 26 with $\kappa = \aleph_0$ gives the following result, which can be seen as a generalization to trees of Theorem 10 in that case:

Corollary 27 ([52, Corollary 20])**.** *For any positive integer k,*

$$\textit{nonspecial tree} \rightarrow \left(\textit{nonspecial tree},\ (\omega+1)_k\right)^2 .$$

The following result is a generalization to trees of the Baumgartner-Hajnal Theorem (Theorem 13):

Theorem 28 ([52, Theorem 1])**.** *For all $\alpha < \omega_1$ and $k < \omega$, we have*

$$\textit{nonspecial tree} \rightarrow (\alpha)^2_k .$$

Our main focus will be the following partition relation for trees, which is a generalization to trees of the Balanced Baumgartner-Hajnal-Todorcevic Theorem (Theorem 8):

Theorem 29 (Main Theorem)**.** *Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then*

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree} \rightarrow (\kappa+\xi)^2_k .$$

We shall return to the Main Theorem in chapter 4.

What about partitioning triples from nonspecial trees?

Again taking into account the limitation provided by Theorem 34, we support Jones' generalization to trees of the Erdős-Rado Conjecture for triples, namely, a *yes* answer to the following:

Question 1. *Do we have, for all $\alpha < \omega_1$ and $n < \omega$,*

$$\textit{nonspecial tree} \rightarrow (\alpha, n)^3 ?$$

We have independently proven the results contained in the following theorem of Jones, whose two parts are (respectively) generalizations to trees of Theorems 15 and 16:

Theorem 30 ([24])**.**

$$\textit{nonspecial tree} \rightarrow (\omega+\omega+1, 4)^3 ;$$

$$\textit{nonspecial tree} \rightarrow (\omega+m, n)^3 \qquad \textit{for all } m, n < \omega.$$

1.6 A Crucial Counterexample

The previous section suggests a general conjecture: Can we prove a theorem to the effect that any partition relation that is true for a successor cardinal κ^+ remains true when the cardinal κ^+ is replaced by an arbitrary non-κ-special tree?

Unfortunately, the answer is *no*, at least not without avoiding the natural axiom MA + ¬CH, as we see from the following contrasting results:

Theorem 31 (Todorcevic, see [15, S42A–B]). *$MA_{\aleph_1}$ implies*

$$\omega_1 \to \left(\omega_1, \omega^2\right)^2.$$

Theorem 32 ([52, Theorem 21]). *MA implies that for any nonspecial tree T such that $|T| = \mathfrak{c}$ and T includes no uncountable chain, we have*

$$T \not\to (\textit{non-special tree, } \omega + 2)^2.$$

As we mentioned earlier, Kurepa showed [33, Theorem 1] [34, p. 236] that there *does* exist a nonspecial tree with no uncountable chain, namely $w\mathbb{Q}$ (and its variant $\sigma\mathbb{Q}$), the collection of all well-ordered subsets of $\mathbb{Q}$, ordered by end-extension. Since $|w\mathbb{Q}| = \mathfrak{c}$, this tree witnesses:

Corollary 33 ([52, Corollary 22]). *MA implies*

$$\textit{non-special tree } \not\to (\textit{non-special tree, } \omega + 2)^2.$$

So we see that under MA + ¬CH, the positive partition relation on ω_1 given by Theorem 31 does not generalize to non-special trees of height ω_1.

What about aiming for a positive consistency result by avoiding MA + ¬CH, which caused this limitation?

Question 2. *Is it consistent that any partition relation that is true for a successor cardinal κ^+ remains true when the cardinal κ^+ is replaced by an arbitrary non-κ-special tree?*

1.7 A Negative Relation for Triples

Toward the end of section 1.1, we mentioned the negative result

$$\omega_1 \not\to (\omega + 2, \omega)^3.$$

We have not found a proof of this result in the literature,[8] so we offer one here:

[8] Jones [23, p. 1198, (6)(b)], Milner-Prikry [38, p. 488, (1.4)], and Hajnal-Larson [20, p. 142] attribute this result to [13], but the closest statements proven there are

1. the weaker result
$$\omega_1 \not\to (\omega + 2, \omega + 1)^3$$
(p. 472, (97), and also in [28, Theorem 18]), and
2. the equivalent statement for a *real type* rather than ω_1 (Theorem 27 on p. 444).

Theorem 34.

$$\omega_1 \not\to (\omega + 2, \omega)^3 .$$

Proof. Using the method of minimal walks on countable ordinals, Todorcevic has shown [53, Definition 3.2.1, Lemma 3.2.3] that there exists a function

$$\bar{\rho} : [\omega_1]^2 \to \omega$$

with the property that

$$(\forall \alpha < \beta < \gamma < \omega_1) \left[\bar{\rho}(\alpha, \beta) \neq \bar{\rho}(\beta, \gamma)\right].$$

This allows us to define a partition $f : [\omega_1]^3 \to 2$ by setting, for $\langle x, y, z\rangle_< \in [\omega_1]^3$,

$$f\left(\langle x, y, z\rangle_<\right) = \begin{cases} 0 & \text{if } \bar{\rho}\,\langle x, y\rangle < \bar{\rho}\,\langle y, z\rangle\,; \\ 1 & \text{if } \bar{\rho}\,\langle x, y\rangle > \bar{\rho}\,\langle y, z\rangle\,. \end{cases}$$

Suppose f has a 0-homogeneous sequence

$$A = \langle a_0, a_1, \ldots, a_\omega, a_{\omega+1}\rangle_< \in [\omega_1]^{\omega+2} .$$

Then for all $n < \omega$ we have $\langle a_n, a_{n+1}, a_{n+2}\rangle_< \in [A]^3$, so that $f\,\langle a_n, a_{n+1}, a_{n+2}\rangle = 0$ and $\bar{\rho}\,\langle a_n, a_{n+1}\rangle < \bar{\rho}\,\langle a_{n+1}, a_{n+2}\rangle$. This gives us an infinite increasing sequence of natural numbers

$$\langle \bar{\rho}\,\langle a_n, a_{n+1}\rangle : n < \omega\rangle .$$

But for all $n < \omega$ we also have

$$\langle a_n, a_{n+1}, a_\omega\rangle_< , \langle a_{n+1}, a_\omega, a_{\omega+1}\rangle_< \in [A]^3,$$

so that $f\,\langle a_n, a_{n+1}, a_\omega\rangle = f\,\langle a_{n+1}, a_\omega, a_{\omega+1}\rangle = 0$ and

$$\bar{\rho}\,\langle a_n, a_{n+1}\rangle < \bar{\rho}\,\langle a_{n+1}, a_\omega\rangle < \bar{\rho}\,\langle a_\omega, a_{\omega+1}\rangle .$$

But this means $\bar{\rho}\,\langle a_\omega, a_{\omega+1}\rangle$ would have to be a natural number greater than every number of an infinite increasing sequence, which is impossible.

On the other hand, suppose f has a 1-homogeneous sequence

$$B = \langle a_0, a_1, \ldots\rangle_< \in [\omega_1]^\omega .$$

Then for all $n < \omega$ we have $\langle a_n, a_{n+1}, a_{n+2}\rangle_< \in [B]^3$, so that $f\,\langle a_n, a_{n+1}, a_{n+2}\rangle = 1$ and $\bar{\rho}\,\langle a_n, a_{n+1}\rangle > \bar{\rho}\,\langle a_{n+1}, a_{n+2}\rangle$. This gives us an infinite decreasing sequence of natural numbers

$$\langle \bar{\rho}\,\langle a_n, a_{n+1}\rangle : n < \omega\rangle ,$$

which is impossible. ∎

Chapter 2

Notation

Our set-theoretic notation and terminology will generally follow standard conventions, such as in [12, 21, 25, 29, 51, 54]. For clarity and definiteness, and in some cases to resolve conflicts between the various texts, we state the following:

For cardinals ν and κ, where $\nu \geq 2$ and κ is infinite, we define[1]

$$\nu^{<\kappa} = \sup_{\mu<\kappa} \nu^{\mu},$$

where the exponentiation is *cardinal exponentiation*, and the supremum is taken over *cardinals* $\mu < \kappa$.

Following [4], we define[2] $\log \kappa$ (for an infinite cardinal κ) to be the smallest cardinal τ such that $2^{\tau} \geq \kappa$. So for any ordinal ξ, we have

$$\xi < \log \kappa \iff 2^{|\xi|} < \kappa \iff m^{|\xi|} < \kappa \text{ for any finite } m,$$

and in particular, the hypothesis on ξ in the Main Theorem 29 can be stated as $\xi < \log \kappa$.

If $\mathcal{A} \subseteq \mathcal{P}(Z)$ is any set algebra (field of sets) over some set Z, then we follow the convention in [12, p. 171, Definition 29.5(i),(ii)], [25, Section 13.1], and [4] that a sub-collection $I \subseteq \mathcal{A}$ can be an ideal in $\mathcal{A}$ even if $Z \in I$ (so that $I = \mathcal{A}$). If, in fact, $Z \notin I$, then the ideal is called *proper*. A similar allowance is made in the definition of a filter. This will allow us to define ideals and their corresponding filters without verifying that they are proper.

Recall that for any ideal $\mathcal{I}$ in a set algebra $\mathcal{A} \subseteq \mathcal{P}(Z)$, we define:

$$\mathcal{I}^{+} = \mathcal{A} \setminus \mathcal{I} = \{X \in \mathcal{A} : X \notin \mathcal{I}\}$$
$$\mathcal{I}^{*} = \{X \in \mathcal{A} : Z \setminus X \in \mathcal{I}\}$$

$\mathcal{I}^{*}$ is called the *filter dual to* $\mathcal{I}$. $\mathcal{I}^{+}$ is called the *co-ideal* corresponding to $\mathcal{I}$, and sets in $\mathcal{I}^{+}$ are said to be $\mathcal{I}$*-positive*.

We shall always assume T is a tree with order relation $<_T$. Trees will be assumed to have a root,

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[1] Some older texts use $\nu^{\overset{\kappa}{\smile}}$ instead of $\nu^{<\kappa}$, such as [12], [51], and [52].

[2] In [12], this would be denoted $L_3(\kappa)$.

which we denote $\emptyset$. If T is a tree and X is any set of ordinals, then we define

$$T \restriction X = \{t \in T : \mathrm{ht}_T(t) \in X\}.$$

Following [51, p. 239], "Every subset of a tree T will also be considered as a *subtree* of T." This is also as in [25, p. 27]. That is, unlike in [29, Definition III.5.3], we do not require our subtrees to be *downward closed*.

We use *node* as a synonym for element of a tree, following [25, p. 27], [29, p. 204], [54, Definition 2.2.1], and implicitly [21, p. 244], but unlike [51, p. 240] where *node* has a different meaning.

For any tree T, a *limit node* of T is a node whose height is a limit ordinal,[3] while a *successor node* is one whose height is a successor ordinal.

Following Kunen's notation in [29, Definition III.5.1], we shall use $t\downarrow$ (rather than $\hat{t}$ or $\mathrm{pred}(t)$ or $\mathrm{pr}(t)$) for the set of predecessors of the node $t \in T$, and $t\uparrow$ (rather than T^t) for the cone above t. When discussing diagonal unions, it will be crucial that $t\uparrow$ be defined so as *not* to include t. However, as we shall see later, it will be convenient to make an exception for the cone above the root node $\emptyset$, to allow the root to be in the "cone above" *some* node.[4]

Our notation for partition relations on trees (and on partially ordered sets in general) is based on [52], which generalizes the usual Erdős-Rado notation for linear orders as follows:

Suppose $\langle P, <_P\rangle$ is any partial order. If α is any ordinal, we write $[P]^\alpha$ to denote the set of all linearly ordered chains in P of order-type α. If μ is any cardinal and α is any ordinal, the statement

$$P \to (\alpha)^2_\mu$$

means: For any colouring (partition function) $c : [P]^2 \to \mu$, there is a chain $X \in [P]^\alpha$ that is c-homogeneous, that is, $c''[X]^2 = \{\chi\}$ for some colour $\chi < \mu$.

If T is a tree and $c : [T]^2 \to \mu$ is a colouring, where μ is some cardinal, and $\chi < \mu$ is some ordinal (colour), and $t \in T$, we define

$$c_\chi(t) = \{s <_T t : c\{s,t\} = \chi\} \subseteq t\downarrow.$$

For two subsets $A, B \subseteq T$, we shall write $A <_T B$ to mean: for all $a \in A$ and $b \in B$ we have $a <_T b$. In that case, the set $A \otimes B$ denotes

$$\{\{a,b\} : a \in A, b \in B\},$$

which is a subset of $[T]^2$.

[3] Limit nodes correspond to the limit points of T, when we give T the *tree topology*, as we describe later in footnote 6 on page 24.

[4] Similar to [6, p. 8, footnote 1], the root node is "an annoyance when dealing with diagonal unions".

Chapter 3

A Theory of Stationary Trees

In this chapter, we discuss how some standard concepts that are defined on ordinals, such as regressive functions, normal ideals, diagonal unions, and stationary sets, can be generalized to nonspecial trees.

3.1 The Ideal of Special Subtrees of a Tree

Suppose we fix an infinite cardinal κ and a tree of height κ^+. What is the correct analogue in T of the ideal of bounded sets in κ^+? What is the correct analogue in T of the ideal of nonstationary sets in κ^+?

As an analogue to the ideal of bounded sets in κ^+, we consider the collection of κ-special subtrees of T:

Definition 7. *Let T be a tree of height κ^+. We say that $U \subseteq T$ is a* κ-special subtree of T *if U can be written as a union of $\leq \kappa$ many antichains. That is, U is a κ-special subtree of T if*

$$U = \bigcup_{\alpha<\kappa} A_\alpha,$$

where each $A_\alpha \subseteq T$ is an antichain, or equivalently, if

$$\exists f : U \to \kappa \, (\forall t, u \in U) \, [t <_T u \implies f(t) \neq f(u)] \, .$$

The collection of κ-special subtrees of T is clearly a κ^+-complete ideal on T, and it is proper iff T is itself non-κ-special.

The cardinal κ^+ itself is an example of a non-κ-special tree of height κ^+. Letting $T = \kappa^+$, we see that the κ-special subtrees of κ^+ are precisely the bounded subsets of κ^+, supporting the choice of analogue.

The next important concept on cardinals that we should like to generalize to trees is the concept of club, stationary, and nonstationary sets. The problem is that we cannot reasonably define a club subset of a tree in a way that is analogous to a club subset of a cardinal.[1] Instead,[2] we recall the alternate characterization of stationary and nonstationary subsets given by Neumer in [40]:

Theorem 35 (Neumer's Theorem)**.** *For a regular uncountable cardinal λ, and a set $X \subseteq \lambda$, the following are equivalent:*

1. *X intersects every club set of λ;*
2. *For every regressive function $f : X \to \lambda$, there is some $\alpha < \lambda$ such that $f^{-1}(\alpha)$ is unbounded below λ. (In the terminology of diagonal unions: $X \notin \nabla \mathcal{I}$, where $\mathcal{I}$ is the ideal of bounded subsets of λ.)*

We shall use this characterization to motivate similar definitions on trees. First, a few preliminaries:

3.2 Regressive Functions and Diagonal Unions on Trees

We begin by formalizing the following definition, as mentioned in chapter 2:

Definition 8. *For any tree T and node $t \in T$, we define:*

$$t\downarrow = \{s \in T : s <_T t\}$$
$$t\uparrow = \begin{cases} \{s \in T : t <_T s\} & \text{if } t \neq \emptyset \\ T & \text{if } t = \emptyset. \end{cases}$$

Following immediately from the definition is:

Lemma 36. *For any $A \subseteq T$ and $t \in T$ we have:*

$$A \cap t\uparrow = \begin{cases} \{s \in A : t <_T s\} & \text{if } t \neq \emptyset \\ A & \text{if } t = \emptyset. \end{cases}$$

We now define the diagonal union of subsets of a tree, indexed by nodes of the tree. This is a generalization of the corresponding definition for subsets of a cardinal.

Definition 9. *Let T be a tree. For a collection of subsets of T indexed by nodes of T, i.e.*

$$\langle A_t \rangle_{t \in T} \subseteq \mathcal{P}(T),$$

we define its diagonal union *to be*

$$\nabla_{t \in T} A_t = \bigcup_{t \in T} (A_t \cap t\uparrow).$$

[1] A natural attempt would be to consider the collection of *closed cofinal* subsets of a tree. The problem is that this collection is not necessarily a filter base, that is, it is not necessarily *directed*. For example: Consider $\sigma\mathbb{Q}$ to be the collection of all (nonempty) *bounded* well-ordered sequences of rationals, ordered by end-extension. This is a nonspecial tree, as mentioned in section 1.4. Define the two sets

$$C_1 = \{s \in \sigma\mathbb{Q} : \sup(s) \in (n, n+1] \text{ for some even integer } n\};$$
$$C_2 = \{s \in \sigma\mathbb{Q} : \sup(s) \in (n, n+1] \text{ for some odd integer } n\}.$$

Both C_1 and C_2 are closed cofinal subsets of $\sigma\mathbb{Q}$, but $C_1 \cap C_2$ is empty (and therefore not cofinal in $\sigma\mathbb{Q}$).

[2] As an alternative, Professor Frank Tall recommends investigating the idea of *stationary coding sets* on $\mathcal{P}_\kappa(\lambda)$, introduced in [55]. This requires further investigation.

Note that we use ∇, rather than $\sum$ used by some texts such as [21].

The following lemma supplies some elementary observations about the diagonal union operation. They all reflect the basic intuition that when taking the diagonal union of sets A_t, the only part of each A_t that contributes to the result is the part within $t\uparrow$.

Lemma 37. *For any tree T and any collection*

$$\langle A_t \rangle_{t\in T} \subseteq \mathcal{P}(T),$$

we have:

$$\nabla_{t\in T} A_t = \left\{ s \in T : s \in A_\emptyset \cup \bigcup_{t<_T s} A_t \right\}; \tag{$*$}$$

$$\nabla_{t\in T} A_t = \nabla_{t\in T} \left(A_t \cap t\uparrow\right);$$

$$\nabla_{t\in T} A_t = \nabla_{t\in T} \left(A_t \cup t\downarrow \cup (\{t\} \setminus \{\emptyset\})\right);$$

$$\nabla_{t\in T} A_t = \nabla_{t\in T} \left(A_t \cup (T \setminus t\uparrow)\right);$$

$$\nabla_{t\in T} A_t = \nabla_{t\in T} \left(A_t \cup X_t\right), \text{ where each } X_t \subseteq T \setminus t\uparrow;$$

$$\nabla_{t\in T} A_t = \nabla_{t\in T} \left(A_t \setminus X_t\right), \text{ where each } X_t \subseteq T \setminus t\uparrow;$$

$$\nabla_{t\in T} A_t = \nabla_{t\in T} \left(\bigcup_{s \leq_T t} A_s \right);$$

$$\nabla_{t\in T} A_t = \nabla_{t\in T} \left(A_t \setminus \bigcup_{s <_T t} A_s \right).$$

Proof of $(*)$.

$$\begin{aligned}
\nabla_{t\in T} A_t &= \bigcup_{t\in T} \left(A_t \cap t\uparrow\right). \\
&= \{s \in T : (\exists t \in T)\,[s \in A_t \cap t\uparrow]\} \\
&= \{s \in T : (\exists t \in T)\,[s \in A_t \text{ and } (t <_T s \text{ or } t = \emptyset)]\} \\
&= \{s \in T : s \in A_\emptyset \text{ or } (\exists t <_T s)\, s \in A_t\} \\
&= \left\{ s \in T : s \in A_\emptyset \cup \bigcup_{t<_T s} A_t \right\}
\end{aligned}$$

□

Lemma 38. *For any tree T, if the collections*

$$\langle A_t \rangle_{t\in T}, \langle B_t \rangle_{t\in T} \subseteq \mathcal{P}(T)$$

are such that for all $t \in T$ we have $A_t \subseteq B_t$, then

$$\bigtriangledown_{t \in T} A_t \subseteq \bigtriangledown_{t \in T} B_t.$$

Lemma 39. *For any tree T, any index set J, and collections*

$$\left\langle A_t^j \right\rangle_{j \in J, t \in T} \subseteq \mathcal{P}(T),$$

we have

$$\bigcup_{j \in J} \left(\bigtriangledown_{t \in T} A_t^j \right) = \bigtriangledown_{t \in T} \left(\bigcup_{j \in J} A_t^j \right).$$

Proof.

$$\begin{aligned} \bigcup_{j \in J} \left(\bigtriangledown_{t \in T} A_t^j \right) &= \bigcup_{j \in J} \left(\bigcup_{t \in T} \left(A_t^j \cap t\uparrow \right) \right) \\ &= \bigcup_{t \in T} \left(\bigcup_{j \in J} \left(A_t^j \cap t\uparrow \right) \right) \\ &= \bigcup_{t \in T} \left(\left(\bigcup_{j \in J} A_t^j \right) \cap t\uparrow \right) \\ &= \bigtriangledown_{t \in T} \left(\bigcup_{j \in J} A_t^j \right) \end{aligned}$$

□

Definition 10. *Let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal. We define*

$$\bigtriangledown \mathcal{I} = \left\{ \bigtriangledown_{t \in T} A_t : \langle A_t \rangle_{t \in T} \subseteq \mathcal{I} \right\}.$$

Some easy facts about $\bigtriangledown \mathcal{I}$:

Lemma 40. *If $\mathcal{I}$ is any ideal on T, then $\mathcal{I} \subseteq \bigtriangledown \mathcal{I}$, and $\bigtriangledown \mathcal{I}$ is also an ideal, though not necessarily proper. Furthermore, for any cardinal λ, if $\mathcal{I}$ is λ-complete, then so is $\bigtriangledown \mathcal{I}$.*

Notice that the statement $\mathcal{I} \subseteq \bigtriangledown \mathcal{I}$ of Lemma 40 relies crucially on our earlier convention that $\emptyset \in \emptyset\uparrow$. Otherwise any set containing the root would never be in $\bigtriangledown \mathcal{I}$.

Lemma 41. *If $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{P}(T)$ are two ideals such that $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $\bigtriangledown \mathcal{I}_1 \subseteq \bigtriangledown \mathcal{I}_2$.*

Definition 11 ([49, Section 1])**.** *Let $X \subseteq T$. A function $f : X \to T$ is* regressive *if*

$$(\forall t \in X \setminus \{\emptyset\})\, f(t) <_T t.$$

Definition 12 (cf. [6, p. 7])**.** *Let $X \subseteq T$, and let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal on T. A function $f : X \to T$ is called $\mathcal{I}$-*small *if*

$$(\forall t \in T) \left[f^{-1}(t) \in \mathcal{I} \right].$$

In words, a function is $\mathcal{I}$-small iff it is constant only on $\mathcal{I}$-sets. A function is not $\mathcal{I}$-small iff it is constant on some $\mathcal{I}^+$-set.

Lemma 42 (cf. [6, p. 9])**.** *Let T be a tree, and let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal on T. Then*

$$\bigtriangledown \mathcal{I} = \{X \subseteq T : \exists\ \mathcal{I}\text{-small regressive } f : X \to T\}.$$

Proof.

$\subseteq$ Let $X \in \bigtriangledown \mathcal{I}$. Then we can write

$$X = \bigtriangledown_{t \in T} X_t = \bigcup_{t \in T} (X_t \cap t\!\uparrow),$$

where each $X_t \in \mathcal{I}$. Define $f : X \to T$ by setting, for each $s \in X$, $f(s) = t$, where we choose some t such that $s \in X_t \cap t\!\uparrow$. It is clear that f is regressive. Furthermore, for any $t \in T$,

$$f^{-1}(t) \subseteq X_t \in \mathcal{I},$$

so $f^{-1}(t) \in \mathcal{I}$, showing that f is $\mathcal{I}$-small.

$\supseteq$ Let $X \subseteq T$, and fix an $\mathcal{I}$-small regressive function $f : X \to T$. For each $t \in T$, define

$$X_t = f^{-1}(t).$$

Since f is $\mathcal{I}$-small, each $X_t \in \mathcal{I}$. Since f is regressive, we have $X_t \subseteq t\!\uparrow$ for each $t \in T$. We then have

$$\begin{aligned} X &= \bigcup_{t \in T} f^{-1}(t) \\ &= \bigcup_{t \in T} X_t \\ &= \bigcup_{t \in T} (X_t \cap t\!\uparrow) \\ &= \bigtriangledown_{t \in T} X_t \in \bigtriangledown \mathcal{I}. \end{aligned}$$

□

Notice that in the proof of Lemma 42, the special treatment of $\emptyset$ in the definition of $\emptyset\!\uparrow$ corresponds to the exclusion of $\emptyset$ from the requirement that $f(t) <_T t$ in the definition of regressive function.

Taking complements, we have:

Corollary 43. *For any ideal $\mathcal{I} \subseteq \mathcal{P}(T)$, we have*

$$\left(\bigtriangledown \mathcal{I}\right)^+ = \left\{X \subseteq T : (\forall \text{ regressive } f : X \to T)\,(\exists t \in T)\left[f^{-1}(t) \in \mathcal{I}^+\right]\right\}.$$

In words, a set X is $(\bigtriangledown \mathcal{I})$-positive iff every regressive function on X is constant on an $\mathcal{I}^+$-set.

Corollary 44. *For any ideal $\mathcal{I} \subseteq \mathcal{P}(T)$, the following are equivalent:*

1. *$\mathcal{I}$ is closed under diagonal unions, that is, $\bigtriangledown \mathcal{I} = \mathcal{I}$;*

2. *If $X \in \mathcal{I}^+$, and $f : X \to T$ is a regressive function, then f must be constant on some $\mathcal{I}^+$-set, that is, $(\exists t \in T) f^{-1}(t) \in \mathcal{I}^+$.*

Definition 13. *An ideal $\mathcal{I}$ on T is* normal *if it is closed under diagonal unions (that is, $\bigtriangledown \mathcal{I} = \mathcal{I}$), or equivalently, if every regressive function on an $\mathcal{I}^+$ set must be constant on an $\mathcal{I}^+$ set.*

A natural question arises: For a given ideal, how many times must we iterate the diagonal union operation $\bigtriangledown$ before the operation stabilizes and we obtain a normal ideal? In particular, when is $\bigtriangledown$ idempotent? The following lemma gives us a substantial class of ideals for which the answer is *one*, and this will be a useful tool in later proofs:

Lemma 45 (Idempotence Lemma)**.** *Let $\lambda = \mathrm{ht}(T)$, and suppose λ is any cardinal. If $\mathcal{I}$ is a λ-complete ideal on T, then $\bigtriangledown \bigtriangledown \mathcal{I} = \bigtriangledown \mathcal{I}$, that is, $\bigtriangledown \mathcal{I}$ is normal.*

Proof. $\bigtriangledown \mathcal{I} \subseteq \bigtriangledown \bigtriangledown \mathcal{I}$ is always true, so we must show $\bigtriangledown \bigtriangledown \mathcal{I} \subseteq \bigtriangledown \mathcal{I}$. Let $X \in \bigtriangledown \bigtriangledown \mathcal{I}$. We must show $X \in \bigtriangledown \mathcal{I}$.

As $X \in \bigtriangledown \bigtriangledown \mathcal{I}$, we can write

$$X = \bigtriangledown_{t \in T} A_t,$$

where each $A_t \in \bigtriangledown \mathcal{I}$. For each $t \in T$, we can write

$$A_t = \bigtriangledown_{s \in T} B_t^s,$$

where each $B_t^s \in \mathcal{I}$.

Notice that for each $t \in T$, the only part of A_t that contributes to X is the part within $t\uparrow$. For each $s, t \in T$, the only part of B_t^s that contributes to A_t is the part within $s\uparrow$. We therefore have:

- If s and t are incomparable in T, we have $s\uparrow \cap\, t\uparrow = \emptyset$, so B_t^s does not contribute anything to X;
- If $t \leq_T s$ then $s\uparrow \cap\, t\uparrow = s\uparrow$, so the only part of B_t^s that contributes to X is within $s\uparrow$;
- If $s \leq_T t$ then $s\uparrow \cap\, t\uparrow = t\uparrow$, so the only part of B_t^s that contributes to X is within $t\uparrow$.

With this in mind, we collect the sets B_t^s whose contribution to X lies within any $r\uparrow$. We define, for each $r \in T$,

$$D_r = \bigcup_{t \leq_T r} B_t^r \cup \bigcup_{s \leq_T r} B_r^s.$$

Since $\mathcal{I}$ is λ-complete and each r has height $< \lambda$, it is clear that $D_r \in \mathcal{I}$.

Claim 45.1. *We have*

$$X = \bigtriangledown_{r \in T} D_r.$$

Proof.

$$
\begin{aligned}
X &= \bigtriangledown_{t\in T} A_t \\
&= \bigtriangledown_{t\in T} \bigtriangledown_{s\in T} B_t^s \\
&= \bigtriangledown_{t\in T} \bigcup_{s\in T} (B_t^s \cap s\uparrow) \\
&= \bigcup_{t\in T} \left[\bigcup_{s\in T} (B_t^s \cap s\uparrow) \cap t\uparrow \right] \\
&= \bigcup_{t\in T} \bigcup_{s\in T} (B_t^s \cap s\uparrow \cap t\uparrow) \\
&= \bigcup_{t,s\in T} (B_t^s \cap s\uparrow \cap t\uparrow) \\
&= \bigcup_{\substack{t,s\in T\\ t\leq_T s}} (B_t^s \cap s\uparrow) \cup \bigcup_{\substack{t,s\in T\\ s\leq_T t}} (B_t^s \cap t\uparrow) \\
&= \bigcup_{r\in T} \left[\bigcup_{t\leq_T r} (B_t^r \cap r\uparrow) \cup \bigcup_{s\leq_T r} (B_r^s \cap r\uparrow) \right] \\
&= \bigcup_{r\in T} \left[\left(\bigcup_{t\leq_T r} B_t^r \cup \bigcup_{s\leq_T r} B_r^s \right) \cap r\uparrow \right] \\
&= \bigcup_{r\in T} (D_r \cap r\uparrow) \\
&= \bigtriangledown_{r\in T} D_r. \qquad \square
\end{aligned}
$$

It follows that $X \in \bigtriangledown \mathcal{I}$, as required. $\square$

3.3 The Ideal of Nonstationary Subtrees of a Tree

Armed with Neumer's characterization of stationary (and nonstationary) subsets of a cardinal in terms of diagonal unions (Theorem 35), we now explore an analogue for trees of this concept, using the new concepts we have introduced in the previous section:

Definition 14. *Let $B \subseteq T$, where T is a tree of height κ^+. We say that B is a* nonstationary subtree of T *if we can write*

$$B = \bigtriangledown_{t\in T} A_t,$$

*where each A_t is a κ-special subtree of T. We may, for emphasis, refer to B as κ-*nonstationary*. If B cannot be written this way, then B is a* stationary subtree of T*.*

We define NS_κ^T to be the collection of nonstationary subtrees of T. That is, NS_κ^T is the diagonal union of the ideal of κ-special subtrees of T. (The subscript κ is for emphasis and may sometimes be omitted.)

Our definitions here are new, and in particular are different from Todorcevic's earlier use of I_T in [49] and NS_T in [52]. Todorcevic defines NS_T as an ideal on the cardinal κ^+, consisting of subsets of κ^+ that are said to be nonstationary *in* or *with respect to* T, while we define NS_κ^T as an ideal on the tree T itself, consisting of sets that are nonstationary *subsets of* T. For any set $X \subseteq \kappa^+$, the statement $T \restriction X \in NS_\kappa^T$ in our notation means the same thing as $X \in NS_T$ of [52]. However, our definitions will allow greater flexibility in stating and proving the relevant results. In particular, we can discuss the membership of arbitrary subsets of the tree in the ideal NS_κ^T, rather than only those of the form $T \restriction X$ for some $X \subseteq \kappa^+$.

In the case that $T = \kappa^+$, the fact that NS_κ^T is identical to the collection of nonstationary sets in the usual sense (that is, sets whose complements include a club subset of κ^+) is Theorem 35 (Neumer's Theorem), so the analogue is correct. In fact, more can be said about the analogue: In what may be historically the first use[3] of the word *stationary* (actually, the French word *stationnaire*) in the context of regressive functions, Gérard Bloch [7] *defines* a set $A \subseteq \omega_1$ to be stationary if every regressive function on A is constant on an uncountable set, and then states as a *theorem* that a set is stationary iff its complement includes no club subset, rather than using the latter characterization as the definition of stationary as would be done nowadays (cf. [49, p. 251], [6, Prop. I.2.1(i)]). So the extension to stationary subtrees of a tree really is a direct generalization of the original definition of stationary subsets of a cardinal!

The following lemma collects facts about NS_κ^T that follow easily from Lemma 40:

Lemma 46. *Fix a tree T of height κ^+. Then every κ-special subtree of T is a nonstationary subtree. Furthermore, NS_κ^T is a κ^+-complete ideal on T.*

The converse of the first conclusion of Lemma 46 is false. In the special case where T is just the cardinal κ^+, there exist unbounded nonstationary subsets of κ^+ (for example, the set of successor ordinals less that κ^+), so any such set is a nonstationary subtree of κ^+ that is not κ-special. This also means that the ideal of bounded subsets of κ^+ is not normal, so that in general the ideal of κ-special subtrees of a tree T is not a normal ideal. However, we do have the following generalization to trees of Fodor's Theorem:

Theorem 47. *For any tree T of height κ^+, the ideal NS_κ^T is a normal ideal on T.*

Proof. This follows from the Idempotence Lemma (Lemma 45), since the ideal of κ-special subtrees is κ^+-complete. ☐

Theorem 47 tells us that $\nabla NS_\kappa^T = NS_\kappa^T$. Equivalently: If B is a stationary subtree of T, meaning that every regressive function on B is constant on a non-κ-special subtree of T, then in fact every regressive function on B is constant on a stationary subtree of T. So for any tree T of height κ^+, the main tool for extracting subtrees using regressive functions should be the ideal NS_κ^T, rather than the ideal of κ-special subtrees of T.

Theorem 47 is stated without proof as [52, Theorem 13], and the special case for trees of height ω_1 is proven as [49, Theorem 2.2(i)]. The simplicity of our proof, compared to the one in [49], is a result of our new definitions and machinery that we have built up to this point.

The ideal NS_κ^T will be useful if we know that it is proper. When can we guarantee that $T \notin NS_\kappa^T$? The following lemma will be a crucial ingredient in the proof of Theorem 49:

[3]See [35, footnote 214] and [21, p. 105].

Lemma 48. *Let $A \subseteq T$ be an antichain. For each $t \in A$, fix a κ-special subtree $X_t \subseteq t\uparrow$. Then*

$$\bigcup_{t \in A} X_t$$

is also a κ-special subtree of T. That is, a union of κ-special subtrees above pairwise incompatible nodes is also a κ-special subtree.

While Lemma 48 is easily seen to be true, what is significant about it is the precision of its hypotheses. If instead of each X_t being a κ-special subtree we require it to be a union of at most κ *levels* of the tree, then even if we require A to consist of nodes on a single level, we do not get the result that the union of all X_t is a union of κ levels of the tree. So in the development of our theory we cannot replace the ideal of κ-special subtrees with the ideal of subtrees consisting of (at most) κ levels of the tree, even though the latter is also a κ^+-complete ideal on the tree, and may appear to be a reasonable generalization to trees of the concept of bounded subsets of κ^+.

Similarly, if we try to generalize to trees of height a limit cardinal λ rather than κ^+, replacing the ideal of κ-special subtrees with the ideal of subtrees that are unions of strictly fewer than λ antichains, we do not get an analogue of Lemma 48 (even if the height is a regular limit cardinal), and this is why Theorem 49 is not valid for trees of limit-cardinal height.[4]

Obviously, if a tree is special, then all of its subtrees are special and therefore nonstationary. Theorem 49 gives the converse, establishing the significance of using a nonspecial tree as our ambient space. It is a generalization to nonspecial trees of a theorem of Dushnik [10] on successor cardinals[5], which itself was a generalization of Alexandroff and Urysohn's theorem [1] on ω_1.

The proofs in [1] and [10] are substantially different from each other, and each one of them has been generalized to prove theorems for which the other method would not be suitable. The main ingredient in [10] is a cardinality argument, and this is the proof that extends to trees, where the focus will be on counting antichains, as we shall see. On the other hand, the main argument of [1] involves cofinality, and this is the argument that is adaptable to prove Theorem 35 (Neumer's Theorem), but does not extend easily to trees.

The case of Theorem 49 for nonspecial trees of height ω_1 is proven as [49, Theorem 2.4]. The general case is subsumed by [52, Theorem 14], but we present the theorem and its proof here, for several reasons: to indicate the generality of Dushnik's technique as it applies to trees of successor-cardinal height, to isolate this theorem and its proof from the harder portion of [52, Theorem 14] (which we state later as Theorem 52), and to show how the statement of the theorem and its proof are affected by our new terminology and notation.

Theorem 49 (Pressing-Down Lemma for Trees)**.** *Suppose T is a non-κ-special tree. Then NS^T_κ is a proper ideal on T, that is, $T \notin NS^T_\kappa$.*

Proof. Fix a non-κ-special tree T, and suppose $\langle X_t \rangle_{t \in T}$ is any indexed collection of κ-special subtrees

[4] It is possible to extend the definitions of *special* and *nonspecial* to trees with height an arbitrary regular cardinal, as Todorcevic does in [53, Chapter 6]. The essential difficulty is that Theorem 49 doesn't hold for trees of limit-cardinal height, but this is overcome by starting with the characterization $T \notin NS^T_\kappa$ in Theorem 49 as the *definition* of nonspecial, rather than Definition 7. In this exposition, we have chosen to restrict our investigation to trees of successor-cardinal height.

[5] Though he does not say so, Dushnik's proof works for regular limit cardinals as well. Nevertheless, it does not generalize to trees of regular-limit-cardinal height, due to the failure of Lemma 48 in that case, as we have explained.

of T. We shall show that

$$T \neq \bigtriangledown_{t \in T} X_t.$$

We define a sequence of subtrees of T by recursion on $n < \omega$, as follows: Let

$$S_0 = \{\emptyset\},$$

and for $n < \omega$, define

$$S_{n+1} = \bigcup_{t \in S_n} (X_t \cap t\uparrow).$$

Claim 49.1. *For all $n < \omega$, S_n is a κ-special subtree of T.*

Proof. We prove this claim by induction on n. Certainly $S_0 = \{\emptyset\}$ is a κ-special subtree as it contains only one element. Now fix $n < \omega$ and suppose S_n is a κ-special subtree. We need to show that S_{n+1} is κ-special.

Since S_n is κ-special, we can write

$$S_n = \bigcup_{\alpha < \kappa} A_\alpha,$$

where each A_α is an antichain. For each $t \in T$ we know that $X_t \cap t\uparrow$ is a κ-special subtree of $t\uparrow$, so for each $\alpha < \kappa$, Lemma 48 tells us that

$$\bigcup_{t \in A_\alpha} (X_t \cap t\uparrow)$$

is a κ-special subtree of T. We then have

$$S_{n+1} = \bigcup_{\alpha < \kappa} \bigcup_{t \in A_\alpha} (X_t \cap t\uparrow),$$

so that S_{n+1} is a union of κ many κ-special subtrees, and is therefore κ-special, completing the induction.

□

Since T is a non-κ-special tree, and a union of countably many κ-special subtrees is also κ-special, we have

$$T \setminus \bigcup_{n < \omega} S_n \neq \emptyset,$$

so we fix a $<_T$-minimal element s of that set.

Claim 49.2. *We have*

$$s \notin \bigtriangledown_{t \in T} X_t.$$

Proof. By equivalence $(*)$ of Lemma 37, we need to show that

$$s \notin X_\emptyset \cup \bigcup_{t <_T s} X_t.$$

Since $\emptyset \in S_0$, we have $s \neq \emptyset$, so we just need to show that for any $t <_T s$, we have $s \notin X_t$. So suppose $t <_T s$. Since s was minimally not in any S_n, we must have $t \in S_n$ for some $n < \omega$. If s were in X_t, then by definition of S_{n+1} we should have $s \in S_{n+1}$, contradicting the choice of s. So s is not in any relevant X_t, as required. □

We have thus found $s \in T$ that is not in the diagonal union of the κ-special sets X_t, as required to show that $T \notin NS_\kappa^T$. □

What other nonstationary subtrees can we come up with?

Lemma 50. *Let T be any tree of height κ^+, and let $S \subseteq T$ be any subtree. Then the set of isolated points*[6] *of S is a nonstationary subtree of T.*

Proof. Let R be the set of isolated points of S. Define a function $f : R \to T$ by setting, for $t \in R$,

$$f(t) = \sup\left(S \cap t{\downarrow}\right),$$

where the sup is taken along the chain $t{\downarrow} \cup \{t\}$.

For any $t \in R$, t is an isolated point of S, so $S \cap t{\downarrow}$ must be bounded below t, so that $f(t) <_T t$. This shows that f is regressive.

Claim 50.1. *For each $s \in T$, $f^{-1}(s)$ is an antichain.*

Proof. If $t_1 <_T t_2$ are both in R, then $f(t_1) <_T t_1 \leq_T f(t_2)$. □

So R is a diagonal union of antichains, and is therefore a nonstationary subtree of T, as required. □

What do we know about the status of subtrees of the form $T \restriction X$, for some $X \subseteq \kappa^+$, with respect to the ideal NS_κ^T? The following facts are straightforward:

Lemma 51. *Let T be any tree of height κ^+, and let $X, C \subseteq \kappa^+$. Then:*

1. *If $|X| \leq \kappa$ then $T \restriction X$ is a κ-special subtree of T.*
2. *If X is a nonstationary subset of κ^+, then $T \restriction X \in NS_\kappa^T$.*
3. *In particular, the set of successor nodes of T is a nonstationary subtree of T.*
4. *If C is a club subset of κ^+, then $T \restriction C \in (NS_\kappa^T)^*$.*

[6] The topology on T is the *tree topology*, defined by any of the following equivalent formulations:

1. The tree topology has, as its basic open sets, all chains $C \subseteq T$ such that $\mathrm{ht}_T'' C$ is open as a set of ordinals.
2. [43, p. 14] The tree topology has, as its basic open sets, the singleton root $\{\emptyset\}$ as well as all intervals (chains) of the form $(s, t]$ for $s <_T t$ in T.
3. [51, p. 244] The tree topology has, as its basic open sets, all intervals (chains) of the form $(s, t]$ for $s <_T t$ in $T \cup \{-\infty\}$.
4. [29, Definition III.5.15] A set $U \subseteq T$ is open in the tree topology iff for all $t \in U$ with height a limit ordinal,
$$(\exists s <_T t)\left[s{\uparrow} \cap t{\downarrow} \subseteq U\right].$$
5. A point $t \in T$ is a limit point of a set $X \subseteq T$ in the tree topology iff ($\mathrm{ht}_T(t)$ is a limit ordinal and) $X \cap t{\downarrow}$ is unbounded below t.

The fact that the tree topology doesn't have an easily intuitive definition in terms of basic open sets (as seen especially by the awkward semi-open intervals in (2) and (3) above) seems to relate to the fact that there is no obvious order topology on an arbitrary partial order.

5. *If T is a non-κ-special tree and C is a club subset of κ^+, then $T \restriction C \notin NS^T_\kappa$.*

Proof.

1. $T \restriction X$ is a union of $|X|$ antichains.

2. (cf. [49, p. 251]) Let X be a nonstationary subset of κ^+. By Theorem 35 (Neumer's characterization of nonstationary subsets of a cardinal), we can choose a regressive function $f : X \to \kappa^+$ such that $\left|f^{-1}(\alpha)\right| < \kappa^+$ for every $\alpha < \kappa^+$. This induces a regressive function $f_T : T \restriction X \to T$, as follows: For every $t \in T \restriction X$, let $f_T(t) \leq_T t$ be such that

$$\mathrm{ht}_T(f_T(t)) = f(\mathrm{ht}_T(t)).$$

The function f_T is well-defined and regressive, and for each $s \in T$ the set $f_T^{-1}(s)$ is a κ-special subtree by part (1), since it is a subset of $T \restriction f^{-1}(\mathrm{ht}_T(s))$. It follows that $T \restriction X$ is a nonstationary subtree of T, as required.

3. This follows from part (2), since the set of successor ordinals below κ^+ is a nonstationary subset of κ^+. Alternatively, the successor nodes are precisely the isolated points of T, so we can apply Lemma 50 to the whole tree T.

4. We have

$$T \setminus (T \restriction C) = T \restriction \left(\kappa^+ \setminus C\right) \in NS^T_\kappa$$

by part (2), so $T \restriction C$ is the complement of an ideal set and therefore in the filter.

5. By the Pressing-Down Lemma for Trees (Theorem 49), NS^T_κ is a proper ideal on T, so that

$$\left(NS^T_\kappa\right)^* \subseteq \left(NS^T_\kappa\right)^+,$$

and the required result then follows from part (4). ☐

It is a standard textbook theorem (see e.g. [29, Lemma III.6.9]) that for any regular infinite cardinal $\theta < \kappa^+$, the set

$$S^{\kappa^+}_\theta = \left\{\gamma < \kappa^+ : \mathrm{cf}(\gamma) = \theta\right\}$$

is a stationary subset of κ^+. A partial analogue to this theorem for trees is:

Theorem 52 ([52, Theorem 14, (2) $\implies$ (3)])**.** *If T is a non-κ-special tree, then the subtree*

$$T \restriction S^{\kappa^+}_{\mathrm{cf}(\kappa)} = \{t \in T : \mathrm{cf}(\mathrm{ht}_T(t)) = \mathrm{cf}(\kappa)\}$$

is a stationary subtree of T.

Of course, in the case where T has height ω_1 (that is, where $\kappa = \omega$), Theorem 52 provides no new information, because the set of ordinals with countable cofinality is just the set of limit ordinals below ω_1 and is therefore a club subset of ω_1, so that Lemma 51(5) applies. But when $\kappa > \omega$, Theorem 52 provides a nontrivial example of a stationary subtree of T whose complement is not (necessarily) nonstationary.

Chapter 4

The Balanced Baumgartner-Hajnal-Todorcevic Theorem for Trees

4.1 Balanced Baumgartner-Hajnal-Todorcevic Theorem for Trees: Background and Motivation

The remainder of this paper is devoted to our exposition of the Main Theorem, Theorem 29, which we restate here for reference:

Main Theorem. *Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then*

$$\textit{non-}\big(2^{<\kappa}\big)\textit{-special tree } \to (\kappa + \xi)^2_k .$$

The Main Theorem is a generalization to trees of the Balanced Baumgartner-Hajnal-Todorcevic Theorem,[1] which we have stated earlier as Theorem 8 and which we recover by applying the Main Theorem 29 to the cardinal $(2^{<\kappa})^+$, the simplest example of a non-$(2^{<\kappa})$-special tree.

The case of the Main Theorem 29 where $k = 2$ was proven by Todorcevic in [52, Theorem 2]. This was a generalization to trees of the corresponding result for cardinals by Shelah [44, Theorem 6.1].

The Main Theorem 29 is a partial strengthening of the following result of Todorcevic, which is the case $r = 2$ of Theorem 25, and is itself a generalization to trees of the balanced Erdős-Rado Theorem for pairs:

Theorem 53 ([52, Corollary 25])**.** *Let κ be any infinite cardinal. Then for any cardinal $\mu < \mathrm{cf}(\kappa)$, we have*

$$\textit{non-}\big(2^{<\kappa}\big)\textit{-special tree } \to (\kappa + 1)^2_\mu .$$

The Main Theorem 29 strengthens the result of Theorem 53 in the sense of providing a longer ordinal goal: $\kappa + \xi$ (for $\xi < \log \kappa$) instead of $\kappa + 1$. However, this comes at a cost: While Theorem 53 applies

[1] Jean Larson refers to it by that name in [35, p. 312, p. 326].

to any infinite cardinal κ, the Main Theorem 29 applies to regular cardinals only (see section 4.4 for discussion of the singular case); and while Theorem 53 allows any number of colours less than $\mathrm{cf}(\kappa)$, the colouring in the Main Theorem 29 must be finite.

One of the main tools we shall use in our proof of the Main Theorem 29 is the technique of *non-reflecting ideals determined by elementary submodels*. This technique was introduced in [4], where in Sections 1–3 it is used to prove the Balanced Baumgartner-Hajnal-Todorcevic Theorem (our Theorem 8). Those sections of [4] are reproduced almost verbatim in [2]. The basics of the technique are exposed in [37, Section 2], and the method is developed in [20, Sections 3 and 4]. Some history of this technique is described in [35, pp. 312–313]. In our section 4.6 below, we shall explain the technique in detail, while developing a more general form that works for trees rather than cardinals.

4.2 Limitations, Conjectures, and Open Questions

What possibilities are there for further extensions of the Main Theorem 29?

First of all, we notice that the hypothesis that the tree is non-$(2^{<\kappa})$-special is necessary, due to the combination of the following two theorems:

Theorem 54. *Let κ be any infinite cardinal. If T is any κ-special tree, then*

$$T \not\to (\kappa+1, \omega)^2 .$$

This theorem is a generalization to trees of the relation for ordinals given in [54, Theorem 7.1.5]. The special case where $\kappa = \omega$ is given in [16, Theorem 7].

Proof. Let $f : T \to \kappa$ be a κ-specializing map for T. So for any $\{x, y\} \in [T]^2$, we clearly have $f(x) \neq f(y)$. Define a colouring

$$g : [T]^2 \to 2 = \{0, 1\}$$

by setting, for $\{x, y\} \in [T]^2$ with $x <_T y$,

$$g(\{x, y\}) = \begin{cases} 0 & \text{if } f(x) < f(y); \\ 1 & \text{if } f(x) > f(y). \end{cases}$$

Suppose $A \subseteq T$ is a 0-homogeneous chain for g. Then $\langle f(x) : x \in A\rangle$ is a sequence in κ of the same order type as A, so the order type of A cannot be greater than κ.

Suppose $B \subseteq T$ is a 1-homogeneous chain for g. Then $\langle f(x) : x \in B\rangle$ is a decreasing sequence of ordinals, so B cannot be infinite. $\square$

Theorem 55. *Let κ be any infinite regular cardinal, and suppose $2^{<\kappa} > \kappa$. If T is any $(2^{<\kappa})$-special tree, then*

$$T \not\to (\kappa)^2_2$$

Proof. Since $2^{<\kappa} > \kappa$, we can find some $\mu < \kappa$ such that $2^\mu > \kappa$. Using a Sierpiński partition [12, bottom of p. 108] [21, Lemma 9.4] [25, Theorem 15.12], we have

$$2^\mu \not\to \left(\mu^+\right)^2_2 .$$

But $\kappa < 2^\mu$ and $\mu^+ \leq \kappa$, so this implies $\kappa \not\to (\kappa)^2_2$. Then, since κ is regular, [12, Corollary 21.5(iii)] ensures that

$$2^{<\kappa} \not\to (\kappa)^2_2 .$$

Combining a colouring $c : [2^{<\kappa}]^2 \to 2$ witnessing this last negative partition relation with a specializing map $f : [T] \to 2^{<\kappa}$ in the obvious way induces a colouring $c' : [T]^2 \to 2$ with the desired properties. □

Corollary 56. *Let κ be any infinite regular cardinal. If T is any $(2^{<\kappa})$-special tree, then*

$$T \not\to (\kappa + 1, \kappa)^2 .$$

Proof. If $2^{<\kappa} = \kappa$, apply Theorem 54. Otherwise $2^{<\kappa} > \kappa$, so apply Theorem 55. □

Combining this last result with our Main Theorem, we obtain the following equivalence:

Theorem 57. *Let κ be any infinite regular cardinal. If T is any tree, then the following are equivalent:*

1. $T \to (2)^1_{2^{<\kappa}}$
2. $T \to (2^{<\kappa})^1_{2^{<\kappa}}$
3. $T \to (\kappa + 1, \kappa)^2$
4. *For any ordinal ξ such that $2^{|\xi|} < \kappa$, and any natural number k, we have*

$$T \to (\kappa + \xi)^2_k .$$

Proof. Condition (1) is the statement that T is non-$(2^{<\kappa})$-special.

(1) $\iff$ **(2)** Standard.

(1) $\implies$ **(4)** This is the Main Theorem, Theorem 29.

(4) $\implies$ **(3)** By monotonicity of the subscript and the ordinal goals.

(3) $\implies$ **(1)** This is Corollary 56. □

Next we consider: Is there any hope of extending the ordinal goals beyond the ordinal $\kappa+\xi$ (where $\xi < \log \kappa$) of the Main Theorem 29? Can we get a homogeneous chain of order-type $\kappa + \log \kappa$? Alternatively, can we somehow combine the ordinal goals of the Main Theorem 29 with the infinite number of colours in Theorem 53?

When $\kappa = \aleph_0$, both the Main Theorem 29 and Theorem 53 are subsumed by a stronger result of Todorcevic [52, Theorem 1], which we have stated earlier (Theorem 28): For all $\alpha < \omega_1$ and $k < \omega$ we have

$$\text{nonspecial tree } \to (\alpha)^2_k .$$

What about uncountable values of κ?

The following theorem collects various results from [26, Section 3] that limit the possible extensions of our Main Theorem 29 that we can hope to prove without any special axioms:

Theorem 58. *If $V = L$, then:*[2]

1. *If κ is any regular uncountable cardinal that is not weakly compact, then*

$$\kappa^+ \not\to (\kappa + \log\kappa)^2_2 .$$

2. *For any infinite cardinal κ, we have*

$$\kappa^+ \not\to (\kappa + 2)^2_{\log\kappa} .$$

Recall that $V = L$ implies GCH, which in turn implies:

- $2^{<\kappa} = \kappa$;
- $\log\kappa = \kappa^-$ for a successor cardinal κ (where κ^- is the cardinal μ such that $\mu^+ = \kappa$);
- $\log\kappa = \kappa$ for a limit cardinal κ.

So part (1) of Theorem 58 shows that (for regular uncountable cardinals that are not weakly compact) we cannot extend the ordinal goals of the Main Theorem 29 without any special axioms. That is, just as Theorem 8 is described in [20, p. 142], our Main Theorem 29 is the best possible balanced generalization to trees of the Erdős-Rado Theorem for finitely many colours to ordinal goals.

Furthermore, part (2) of Theorem 58 shows that for successor cardinals κ (where $\log\kappa < \kappa = \mathrm{cf}(\kappa)$) we cannot combine the ordinal goals of the Main Theorem 29 with the larger number of colours in Theorem 53.

This leaves open the following questions:

Question 3. *For a regular cardinal $\kappa > \aleph_1$, do we have*[3]

$$\textit{non-}\big(2^{<\kappa}\big)\textit{-special tree } \to (\kappa + \xi)^2_\mu$$

for $\xi, \mu < \log\kappa$ (or even $\mu^+ < \kappa$)?

The simplest case of the above question is when $\mu = \aleph_0$ and $\kappa = \aleph_2$, so that we ask: Does

$$\big(2^{\aleph_1}\big)^+ \to (\omega_2 + 2)^2_{\aleph_0} ?$$

[2] The negative partition relations proved in [26] are actually stronger (the notation follows [26, p. 153]):

For part (1), we have

$$\kappa^+ \not\to [\kappa : \log\kappa]^2_\kappa ,$$

discussed for successor cardinals in [26, p. 161] (this result was proven by Joseph Rebholz) and for inaccessible cardinals that are not weakly compact in [26, Theorem 3.6] (this result was proven by Hans-Dieter Donder).

For part (2), the result for successor cardinals is

$$\kappa^+ \not\to (\kappa : 2)^2_{\log\kappa} ,$$

given in [26, top of p. 163]. For a limit cardinal we have $\log\kappa = \kappa$ (because we have assumed $V = L$), so the result follows from a standard relation $2^\kappa \not\to (3)^2_\kappa$ [13, Theorem 4(ii)], [12, (19.17)], [54, Corollary 2.5.2], [25, Theorem 15.15], [21, Lemma 9.3].

[3] According to [20, top of p. 143], the only known result in this direction is a result of Shelah [45] that

$$\big(2^{<\kappa}\big)^+ \to (\kappa + \mu)^2_\mu$$

(for regular κ) with the assumption that there exists a strongly compact cardinal σ such that $\mu < \sigma \le \kappa$. We conjecture that this result is true when generalized to trees as well.

We conjecture a *yes* answer to the following question, generalizing the corresponding conjecture for cardinals in [26, p. 156]:

Question 4. *If κ is a weakly compact cardinal, do we have, for every $\alpha < \kappa^+$ and $n < \omega$,*

$$\begin{aligned} \textit{non-}\kappa\textit{-special tree } &\to (\alpha)^2_n ? \\ \textit{non-}\kappa\textit{-special tree } &\to (\kappa^n)^2_{\aleph_0} ? \end{aligned}$$

What about aiming for positive consistency results by avoiding $V = L$, which caused the limitations in Theorem 58 above?

For any fixed uncountable cardinal κ, if $2^{<\kappa} = 2^\kappa$, then applying Theorem 53 to the cardinal κ^+ instead of κ subsumes any extensions of the ordinal goals or number of colours that we could anticipate when applying our Main Theorem 29 to κ.

So the question remains:

Question 5. *Are any extensions of the Main Theorem 29 that are precluded when $V = L$ by Theorem 58 consistent with $2^{<\kappa} < 2^\kappa$?*[4]

Singular cardinals are beyond the scope of this discussion. In section 4.4 we shall explain how our method of proof does not provide any results for singular cardinals.

4.3 Examples

Let us consider some examples of regular cardinals κ, to see what the Main Theorem 29 gives us in each case:

Example 59. *Suppose $\kappa = \aleph_0$. Then $2^{<\kappa} = \aleph_0$, and we have, for any natural numbers k and n,*

$$\textit{nonspecial tree } \to (\omega + n)^2_k .$$

Notice that our proof remains valid in this case; nowhere in the proof of the Main Theorem 29 do we require κ to be uncountable. However, as we have mentioned earlier, this case is already subsumed by the stronger Theorem 28 of Todorcevic.

So we focus on uncountable values of κ. The first case where we get something new is:

Example 60. *Let $\kappa = \aleph_1$. Then $2^{<\kappa} = \mathfrak{c}$, but ξ must still be finite, so we have, for any natural numbers k and n,*

$$\textit{non-}\mathfrak{c}\textit{-special tree } \to (\omega_1 + n)^2_k .$$

Example 60 is the simplest example provided by the Main Theorem 29. However, we can (consistently) strengthen Example 60 by replacing ω_1 with any regular cardinal κ such that $2^{<\kappa} = \mathfrak{c}$. For example:

Example 61. *Suppose $\kappa = \mathfrak{p}$ (the pseudo-intersection number). Then by [29, Exercise III.1.38], κ is regular, and by [29, Lemma III.1.26], $2^{<\kappa} = \mathfrak{c}$. So we have, for any natural numbers k and n,*

$$\textit{non-}\mathfrak{c}\textit{-special tree } \to (\mathfrak{p} + n)^2_k .$$

[4] Some partial results in this direction are presented in [26, Section 2], and we conjecture that they can be generalized to trees.

Example 62. *Setting $\kappa = \mathfrak{c}^+$, we have $2^{<\kappa} = 2^{\mathfrak{c}}$, so that for any ordinal*[5] $\xi < \log(\mathfrak{c}^+)$*, and any natural number k, we have*

$$\textit{non-}2^{\mathfrak{c}}\textit{-special tree} \to \left(\mathfrak{c}^+ + \xi\right)^2_k .$$

If we assume CH, then Example 60 becomes (for finite n and k)

$$\text{non-}\aleph_1\text{-special tree} \to (\omega_1 + n)^2_k .$$

If we further assume GCH, then $2^{<\kappa} = \kappa$, so the general statement of the Main Theorem 29 is simplified to

$$\text{non-}\kappa\text{-special tree} \to (\kappa + \xi)^2_k ,$$

where the hypothesis $\xi < \log \kappa$ can be written as $|\xi|^+ < \kappa$. We shall not assume these (or any) extra axioms in the proof of the Main Theorem 29, but if such assumptions[6] help the reader's intuition in following the proof then there is no harm in doing so.

4.4 The Role of Regularity and Discussion of Singular Cardinals

In the statement of the Main Theorem (Theorem 29), why do we require κ to be regular? Where is the regularity of κ used in the proof? Furthermore, where in the proof do we use the fact that the tree is non-$(2^{<\kappa})$-special?

Suppose we fix *any* infinite cardinal κ. How tall must a non-special tree be in order to obtain the homogeneous sets in the conclusion of the Main Theorem 29?

In order to apply the lemmas of section 4.7 (in particular, Lemmas 90(3) and 94), we shall require our tree to be non-ν-special, where ν is an infinite cardinal satisfying $\nu^{<\kappa} = \nu$. So we need to determine: What is the smallest infinite cardinal ν for which $\nu^{<\kappa} = \nu$? It is clear that we must have $\nu \geq 2^{<\kappa}$. What happens if we set $\nu = 2^{<\kappa}$?

The following fact follows immediately from [12, Theorem 6.10(f), first case]:

Theorem 63. *For any regular cardinal κ, we have*

$$\left(2^{<\kappa}\right)^{<\kappa} = 2^{<\kappa}.$$

So for a *regular* cardinal κ, we can set $\nu = 2^{<\kappa}$ to satisfy the requirement $\nu^{<\kappa} = \nu$, so that the non-$(2^{<\kappa})$-special tree in the statement of the Main Theorem 29 is exactly what we need for the proof to work.

What about the case where κ is a *singular* cardinal? It turns out that Theorem 63 is the only consequence of regularity used in the proof of the Main Theorem 29. In fact, the proof of the Main Theorem 29 actually gives the following (apparently) more general version of it, with weaker hypotheses:

[5] In particular, ξ can be any countable ordinal. More generally, we know from [29, Lemma III.1.26] that $\mathfrak{p} \leq \log(\mathfrak{c}^+)$, so that any $\xi < \mathfrak{p}$ will work.

[6] Older papers often assume GCH, or variants of it, when stating related results, due to the lack of good notation for iterated exponentiation [35, p. 218] and for the weak power $2^{<\kappa}$. Even Section 1 of [4] and [2] (though not the subsequent sections, 2 and 3) unnecessarily assumes $2^{<\kappa} = \kappa$.

Theorem 64. *Let ν and κ be infinite cardinals such that $\nu^{<\kappa} = \nu$. Then for any ordinal ξ such that $2^{|\xi|} < \kappa$, and any natural number k, we have*

$$\text{non-}\nu\text{-special tree } \to (\kappa + \xi)^2_k .$$

For a singular cardinal κ, we should be able to find some infinite cardinal ν satisfying the requirement $\nu^{<\kappa} = \nu$, so that we can apply Theorem 64 to a non-ν-special tree for such ν. It would seem to be significant that the κ in the conclusion does not need to be weakened to $\mathrm{cf}(\kappa)$. It is tempting to conclude that we should present Theorem 64 as our main result, since it appears to have broader application than our Main Theorem 29.

In particular, depending on the values of the continuum function, there may be some singular cardinals κ for which the sequence $\{2^\mu : \mu < \kappa\}$ is eventually constant, in which case any such κ would satisfy $\mathrm{cf}(2^{<\kappa}) \geq \kappa$ and $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$. (Of course, this cannot happen under GCH.) For such κ, we can apply Theorem 64 with $\nu = 2^{<\kappa}$, just as we do for regular cardinals.

However, it turns out that the singular case gives no new results, as we shall see presently.

The following fact follows immediately from [12, Theorem 6.10(f), second case]:

Theorem 65. *For any singular cardinal κ and any cardinal $\nu \geq 2$, we have*

$$\left(\nu^{<\kappa}\right)^{<\kappa} = \nu^\kappa.$$

Fixing any singular cardinal κ, suppose we choose some infinite cardinal ν satisfying $\nu^{<\kappa} = \nu$, in order to apply Theorem 64 to a non-ν-special tree. But then we also have (using Theorem 65)

$$2^{<(\kappa^+)} = 2^\kappa \leq \nu^\kappa = \left(\nu^{<\kappa}\right)^{<\kappa} = \nu^{<\kappa} = \nu,$$

so that the non-ν-special tree in Theorem 64 is also non-$(2^{<(\kappa^+)})$-special. Applying the Main Theorem 29 to the regular (successor) cardinal κ^+ gives us a longer homogeneous chain (of order-type $> \kappa^+$) than the one we get when applying Theorem 64 to the original singular cardinal κ, without requiring a taller tree. Thus any result we can get by applying Theorem 64 to a singular cardinal κ is already subsumed by the Main Theorem 29.

This explains how our Main Theorem 29 is the optimal statement of the result; nothing is gained by attempting to state a more general result that includes singular cardinals.

4.5 From Trees to Partial Orders

In this section we derive a corollary of our Main Theorem 29, using a result of Todorcevic [52, Section 1] that states that partition relations for nonspecial trees imply corresponding partition relations for partially ordered sets in general.

First, we outline the main result of [52, Section 1]:

Theorem 66. *Let r be any positive integer, let κ and θ be cardinals, and for each $\gamma < \theta$ let α_γ be an ordinal. If every non-κ-special tree T satisfies*

$$T \to (\alpha_\gamma)^r_{\gamma<\theta} , \tag{**}$$

then every partial order P satisfying $P \to (\kappa)^1_\kappa$ also satisfies the above partition relation $(**)$.

Proof. Suppose $\langle P, <_P \rangle$ is any partial order satisfying $P \to (\kappa)^1_\kappa$. Let $\sigma' P$ be the set of well-ordered chains of P with a maximal element, ordered by end-extension ($\sqsubseteq$).

Since $P \to (\kappa)^1_\kappa$, [52, Theorem 9] tells us that $\sigma' P$ is not the union of κ antichains. But $\sigma' P$ is clearly a tree (as it is a collection of well-ordered sets, ordered by end-extension), so this means that $\sigma' P$ is a non-κ-special tree. By the hypothesis of our theorem it follows that $\sigma' P$ satisfies $(**)$.

We now define a function $f : \sigma' P \to P$ by setting, for each $a \in \sigma' P$,

$$f(a) = \max(a).$$

It is clear that

$$f : \langle \sigma' P, \sqsubseteq \rangle \to \langle P, <_P \rangle$$

is an order-homomorphism.[7]

Since $\sigma' P$ satisfies $(**)$ and f is an order-homomorphism from $\sigma' P$ into P, it follows from [52, Lemma 1] that P satisfies $(**)$ as well. This is what we needed to show. ☐

Theorem 67. *Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Let P be a partially ordered set such that $P \to (2^{<\kappa})^1_{2^{<\kappa}}$. Then*

$$P \to (\kappa + \xi)^2_k .$$

Proof. Apply Theorem 66 to the Main Theorem 29. ☐

4.6 Non-Reflecting Ideals Determined by Elementary Submodels

In this section, we consider a fixed tree T and a regular cardinal θ such that $T \in H(\theta)$. We shall consider elementary submodels $N \prec H(\theta)$ with $T \in N$, and use them to create certain algebraic structures on T. Ultimately, for some nodes $t \in T$, we shall use models N to define ideals on $t\downarrow$. We make no assumptions about the height of the tree T at this point.

Lemma 68. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$. Then the collection $\mathcal{P}(T) \cap N$ is a field of sets (set algebra) over the set T.*

Proof. The collection $\mathcal{P}(T) \cap N$ is clearly a collection of subsets of T. Furthermore:

Nonempty Clearly $\emptyset \in \mathcal{P}(T) \cap N$.

Complements Since $T \in N$, by elementarity of N it follows that $T \setminus B \in N$ for any $B \in N$.

Finite unions Suppose $A, B \in \mathcal{P}(T) \cap N$. The set $A \cup B$ is definable from A and B, so by elementarity of N we have $A \cup B \in N$. A union of subsets of T is certainly a subset of T, so we have $A \cup B \in \mathcal{P}(T) \cap N$, as required to show that $\mathcal{P}(T) \cap N$ is a set algebra. ☐

[7] See footnote 6 on page 6.

Lemma 69. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the collection*

$$\{B \subseteq T : B \in N \text{ and } t \in B\}$$

is an ultrafilter in the set algebra $\mathcal{P}(T) \cap N$, and the collection

$$\{B \subseteq T : B \in N \text{ and } t \notin B\}$$

is the corresponding maximal (proper) ideal in the same set algebra.

What we really want are algebraic structures on $t{\downarrow}$ determined by N. So we now consider what happens when we intersect members of N with $t{\downarrow}$:

Definition 15. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Define a collapsing function*

$$\pi_{N,t} : \mathcal{P}(T) \cap N \to \mathcal{P}(t{\downarrow})$$

by setting, for $B \subseteq T$ with $B \in N$,

$$\pi_{N,t}(B) = B \cap t{\downarrow}.$$

We then define the collection

$$\mathcal{A}_{N,t} = \text{range}\,(\pi_{N,t}) = \{B \cap t{\downarrow} : B \in \mathcal{P}(T) \cap N\} = \{B \cap t{\downarrow} : B \in N\} \subseteq \mathcal{P}\,(t{\downarrow})\,.$$

Lemma 70. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$.*

Then the collection $\mathcal{A}_{N,t}$ is a set algebra over the set $t{\downarrow}$, and the collapsing function $\pi_{N,t}$ defines a surjective homomorphism of set algebras

$$\pi_{N,t} : \langle \mathcal{P}(T) \cap N, \cup, \cap, \setminus, \emptyset, T\rangle \to \langle \mathcal{A}_{N,t}, \cup, \cap, \setminus, \emptyset, t{\downarrow}\rangle\,.$$

Proof. We shall show that $\pi_{N,t} : \mathcal{P}(T) \cap N \to \mathcal{P}(t{\downarrow})$ preserves the set-algebra operations:

We have $\pi_{N,t}(\emptyset) = \emptyset \cap t{\downarrow} = \emptyset$, and $\pi_{N,t}(T) = T \cap t{\downarrow} = t{\downarrow}$, as required.

For every $B \in \mathcal{P}(T) \cap N$, we have

$$\begin{aligned}\pi_{N,t}(T \setminus B) &= (T \setminus B) \cap t{\downarrow} \\ &= (T \cap t{\downarrow}) \setminus (B \cap t{\downarrow}) \\ &= t{\downarrow} \setminus \pi_{N,t}(B),\end{aligned}$$

showing that $\pi_{N,t}$ preserves complements.

Let $\mathcal{C} \subseteq \mathcal{P}(T) \cap N$ be any collection whose union is in N. We then have

$$\begin{aligned}\pi_{N,t}\left(\bigcup_{B\in\mathcal{C}} B\right) &= \left(\bigcup_{B\in\mathcal{C}} B\right) \cap t{\downarrow} \\ &= \bigcup_{B\in\mathcal{C}} (B \cap t{\downarrow}) \\ &= \bigcup_{B\in\mathcal{C}} \pi_{N,t}(B),\end{aligned}$$

so that $\pi_{N,t}$ preserves unions.

Preservation of intersections follows from preservation of complements and unions, using De Morgan's laws.

So we have shown that $\pi_{N,t}$ respects the set-algebra operations (there are no relations in the set-algebra structure; only constants and functions), and is therefore a homomorphism of set algebras onto its range, which is $\mathcal{A}_{N,t}$.

The homomorphic image of a set algebra (where the range is a subset of a power-set algebra, with the usual set-theoretic operations) is a set algebra, so it follows that $\mathcal{A}_{N,t}$ is a set algebra over the set $t\downarrow$. $\square$

Definition 16. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. We define the collections*

$$\begin{aligned}\mathcal{G}_{N,t} &= \{\pi_{N,t}(A) : A \in \mathcal{P}(T) \cap N \text{ and } t \notin A\}\\ &= \{A \cap t\downarrow : A \in N \text{ and } t \notin A\} \subseteq \mathcal{A}_{N,t}, \text{ and}\\ \mathcal{G}^*_{N,t} &= \{\pi_{N,t}(B) : B \in \mathcal{P}(T) \cap N \text{ and } t \in B\}\\ &= \{B \cap t\downarrow : B \in N \text{ and } t \in B\} \subseteq \mathcal{A}_{N,t}.\end{aligned}$$

Lemma 71. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the collection $\mathcal{G}_{N,t}$ is a (not necessarily proper) ideal in the set algebra $\mathcal{A}_{N,t}$, and $\mathcal{G}^*_{N,t}$ is the dual filter corresponding to $\mathcal{G}_{N,t}$.*

Proof. The collections $\mathcal{G}_{N,t}$ and $\mathcal{G}^*_{N,t}$ are (respectively) the homomorphic images (under $\pi_{N,t}$) of the ideal and filter in the algebra $\mathcal{P}(T) \cap N$ given by Lemma 69. More explicitly:

It is clear that both $\mathcal{G}_{N,t}$ and $\mathcal{G}^*_{N,t}$ are subcollections of the set algebra $\mathcal{A}_{N,t}$. Furthermore:

Nonempty $\emptyset \in \mathcal{G}_{N,t}$, since $\emptyset = \emptyset \cap t\downarrow$, $\emptyset \in N$, and $t \notin \emptyset$.

Subsets Suppose X and Y are both in $\mathcal{A}_{N,t}$, with $X \subseteq Y$ and $Y \in \mathcal{G}_{N,t}$. We need to show that $X \in \mathcal{G}_{N,t}$.

Since X and Y are both in $\mathcal{A}_{N,t}$, we have $X = \pi_{N,t}(A)$ and $Y = \pi_{N,t}(B)$ for some $A, B \in \mathcal{P}(T) \cap N$. Furthermore, since $Y \in \mathcal{G}_{N,t}$, we can choose B so that $t \notin B$. We then have $A \cap B \in \mathcal{P}(T) \cap N$, and $t \notin A \cap B$, and

$$\pi_{N,t}(A \cap B) = \pi_{N,t}(A) \cap \pi_{N,t}(B) = X \cap Y = X,$$

showing that $X \in \mathcal{G}_{N,t}$, as required.

Finite unions Suppose $X, Y \in \mathcal{G}_{N,t}$. We can choose $A, B \in \mathcal{P}(T) \cap N$ with $t \notin A, B$ such that $\pi_{N,t}(A) = X$ and $\pi_{N,t}(B) = Y$. We then have $A \cup B \in \mathcal{P}(T) \cap N$, and $t \notin A \cup B$, and

$$\pi_{N,t}(A \cup B) = \pi_{N,t}(A) \cup \pi_{N,t}(B) = X \cup Y,$$

and it follows that $X \cup Y \in \mathcal{G}_{N,t}$.

Dual filter Finally, for any $X \in \mathcal{A}_{N,t}$, we have

$$\begin{aligned} X \in \mathcal{G}_{N,t} &\iff X = \pi_{N,t}(A) \text{ for some } A \in \mathcal{P}(T) \cap N \text{ with } t \notin A \\ &\iff t{\downarrow} \setminus X = \pi_{N,t}(T \setminus A) \text{ for some } A \in \mathcal{P}(T) \cap N \text{ with } t \notin A \\ &\iff t{\downarrow} \setminus X = \pi_{N,t}(B) \text{ for some } B \in \mathcal{P}(T) \cap N \text{ with } t \in B \\ &\iff t{\downarrow} \setminus X \in \mathcal{G}^*_{N,t}, \end{aligned}$$

showing that $\mathcal{G}_{N,t}$ and $\mathcal{G}^*_{N,t}$ are dual to each other. □

In general, there is no reason to expect that the homomorphism $\pi_{N,t}$ is injective, as there can be many different subsets of T in the model N that share the same intersection with $t{\downarrow}$. As we shall see later (see Remark 79), this is the new difficulty that arises when generalizing these structures from cardinals to trees. In particular, it may happen that $\pi_{N,t}$ collapses the algebra to the extent that $\mathcal{G}_{N,t}$, which is the image of a maximal proper ideal, is equal to the whole algebra $\mathcal{A}_{N,t}$ rather than a proper ideal in it. That is, there may be some $A \in \mathcal{P}(T) \cap N$ with $t \notin A$ but $\pi_{N,t}(A) = t{\downarrow}$. More generally, there may be some $B \in N$ with $t \in B$, but $B \cap t{\downarrow} \in \mathcal{G}_{N,t}$ because it is equal to $A \cap t{\downarrow}$ for some $A \in N$ with $t \notin A$. We shall need to avoid such combinations of models and nodes, and we shall show later how to do so. In the meantime:

Lemma 72. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then*

$$\mathcal{G}_{N,t} \cup \mathcal{G}^*_{N,t} = \mathcal{A}_{N,t},$$

so that exactly one of the following two alternatives is true:

1. $\mathcal{G}_{N,t} = \mathcal{G}^*_{N,t} = \mathcal{A}_{N,t}$, *or*

2. $\mathcal{G}_{N,t}$ *is a maximal proper ideal in* $\mathcal{A}_{N,t}$, *and* $\mathcal{G}^*_{N,t}$ *is the corresponding ultrafilter, so that* $\mathcal{G}_{N,t} \cap \mathcal{G}^*_{N,t} = \emptyset$.

Proof. This follows from the fact that $\mathcal{G}_{N,t}$ and $\mathcal{G}^*_{N,t}$ are (respectively) the homomorphic images (under $\pi_{N,t}$) of the maximal proper ideal and ultrafilter in the algebra $\mathcal{P}(T) \cap N$ given by Lemma 69. □

For elementary submodels $N \prec H(\theta)$ such that $T \in N$, and nodes $t \in T$, recall that $\mathcal{A}_{N,t} \subseteq \mathcal{P}(t{\downarrow})$ is a set algebra over the set $t{\downarrow}$, and we defined a certain ideal $\mathcal{G}_{N,t} \subseteq \mathcal{A}_{N,t}$. We now consider the ideal on $t{\downarrow}$ (that is, the ideal in the whole power set $\mathcal{P}(t{\downarrow})$) generated by $\mathcal{G}_{N,t}$:

Definition 17. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. We define*

$$I_{N,t} = \{X \subseteq t{\downarrow} : X \subseteq Y \textit{ for some } Y \in \mathcal{G}_{N,t}\}.$$

We explore the properties of $I_{N,t}$:

Lemma 73. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the collection $I_{N,t}$ is a (not necessarily proper) ideal on $t{\downarrow}$, that is, an ideal in the whole power set $\mathcal{P}(t{\downarrow})$.*

Proof. Since I_N consists of all subsets of sets in $\mathcal{G}_N$, where by Lemma 71 $\mathcal{G}_N$ is an ideal in the algebra $\mathcal{A}_N \subseteq \mathcal{P}(t{\downarrow})$, it is clear that I_N is an ideal on $t{\downarrow}$. □

Although we have defined the ideal $I_{N,t}$, we shall be more interested in the corresponding co-ideal, $I^+_{N,t}$.

Lemma 74. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then we have the following facts:*

$$\begin{aligned}\mathcal{A}_{N,t} \cap I_{N,t} &= \mathcal{G}_{N,t}\\ \mathcal{A}_{N,t} \cap I^*_{N,t} &= \mathcal{G}^*_{N,t}\\ \mathcal{A}_{N,t} \cap I^+_{N,t} &= \mathcal{G}^+_{N,t}\end{aligned}$$

It follows that for all $B \in N$, we have:

$$\begin{aligned}B \cap t{\downarrow} \in I_{N,t} &\iff B \cap t{\downarrow} \in \mathcal{G}_{N,t}\\ B \cap t{\downarrow} \in I^*_{N,t} &\iff B \cap t{\downarrow} \in \mathcal{G}^*_{N,t}\\ B \cap t{\downarrow} \in I^+_{N,t} &\iff B \cap t{\downarrow} \in \mathcal{G}^+_{N,t}\end{aligned}$$

Furthermore, we can express the ideal, co-ideal and filter as follows:

$$\begin{aligned}I_{N,t} &= \{X \subseteq t{\downarrow} : X \subseteq A \textit{ for some } A \in N \textit{ with } t \notin A\}\\ I^+_{N,t} &= \{X \subseteq t{\downarrow} : \forall A \in N\,[X \subseteq A \implies t \in A]\}\\ I^+_{N,t} &= \{X \subseteq t{\downarrow} : \forall B \in N\,[t \in B \implies X \cap B \neq \emptyset]\}\\ I^*_{N,t} &= \left\{X \subseteq t{\downarrow} : X \supseteq Y \textit{ for some } Y \in \mathcal{G}^*_{N,t}\right\}\\ I^*_{N,t} &= \{X \subseteq t{\downarrow} : X \supseteq B \cap t{\downarrow} \textit{ for some } B \in N \textit{ with } t \in B\}\end{aligned}$$

Finally, $I_{N,t}$ is a proper ideal (in $\mathcal{P}(t{\downarrow})$) iff $\mathcal{G}_{N,t}$ is a proper ideal (in $\mathcal{A}_{N,t}$).

When considering the ideal $I_{N,t}$, we shall generally want to have $t{\downarrow} \subseteq N$. The following lemma explains why:

Lemma 75. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then:*

1. *If $A \subseteq t{\downarrow}$ and $A \in N$, then $A \in \mathcal{G}_{N,t}$.*

Furthermore, if, in addition to the previous hypotheses, we have $t{\downarrow} \subseteq N$, then:

2. *If $s <_T t$, then $s{\downarrow} \in \mathcal{G}_{N,t}$ and $t{\downarrow} \setminus s{\downarrow} \in \mathcal{G}^*_{N,t}$.*

3. *If $X \subseteq t{\downarrow}$ is not cofinal[8] in $t{\downarrow}$, that is, $X \subseteq s{\downarrow}$ for some $s \in t{\downarrow}$, then $X \in I_{N,t}$. Equivalently, any set in $I^+_{N,t}$ must be cofinal in $t{\downarrow}$.*

4. *For any set $Y \subseteq t{\downarrow}$ and any $s \in t{\downarrow}$, we have*

$$Y \in I^+_{N,t} \iff Y \setminus s{\downarrow} \in I^+_{N,t}.$$

[8]For limit nodes t, *not cofinal in* $t{\downarrow}$ is equivalent to *bounded below* t. However, for successor nodes there is a distinction, as for a successor node t every subset of $t{\downarrow}$ has an upper bound below t, namely the node s such that $t{\downarrow} = s{\downarrow} \cup \{s\}$. In fact the stronger statement is true, that every $X \subseteq t{\downarrow}$ that is *bounded below* t is in $I_{N,t}$, but this requires a separate proof for successor nodes, and successor nodes are made irrelevant by Lemma 77 anyway.

Proof.

1. Since $A \subseteq t{\downarrow}$, we certainly have $t \notin A$. Then, since $A \in N$, we also have $\pi_{N,t}(A) = A \cap t{\downarrow} = A$, so it follows that $A \in \mathcal{G}_{N,t}$.

2. Since $t{\downarrow} \subseteq N$, any $s <_T t$ is in N, and $s{\downarrow}$ is defined from s and T, which are both in N, so by elementarity we have $s{\downarrow} \in N$. Clearly, $s{\downarrow} \subseteq t{\downarrow}$, so (1) gives us $s{\downarrow} \in \mathcal{G}_{N,t}$. Then, the corresponding filter set to $s{\downarrow}$ is $t{\downarrow} \setminus s{\downarrow}$, so it follows that $t{\downarrow} \setminus s{\downarrow} \in \mathcal{G}^*_{N,t}$.

3. We have $X \subseteq s{\downarrow}$, where by part (2) we know $s{\downarrow} \in \mathcal{G}_{N,t}$. It follows by definition of $I_{N,t}$ that $X \in I_{N,t}$.

4. From part (2) and Lemma 74, we have $s{\downarrow} \in \mathcal{G}_{N,t} \subseteq I_{N,t}$. Then Y is equivalent to $Y \setminus s{\downarrow}$ modulo a set from the ideal $I_{N,t}$. □

As mentioned earlier, in order to ensure that our ideals are proper, we want to avoid situations where there may be some $B \in N$ with $t \notin B$ but $B \cap t{\downarrow} = t{\downarrow}$. We shall therefore impose an *eligibility condition*:

Definition 18. *Suppose W is any collection of sets. We say that a node $t \in T$ is W-*eligible *if*

$$\forall B \in W \, [t{\downarrow} \subseteq B \implies t \in B] \, .$$

When W is an elementary submodel N, the eligibility condition can be formulated in several ways, in terms of our structures on $t{\downarrow}$. Particularly useful among the following is condition (10), which states that for an N-eligible node t and any $X \in \mathcal{A}_{N,t}$, we can determine whether or not $X \in \mathcal{G}_{N,t}$ by choosing a single $A \in N$ with $\pi_{N,t}(A) = X$ and checking whether or not $t \in A$, rather than having to check every such A.

Lemma 76. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the following are all equivalent:*

1. *t is N-eligible;*

2. *$\nexists A \in N \, [t{\downarrow} \subseteq A \subseteq T \setminus \{t\}]$;*

3. *$t{\downarrow} \notin \mathcal{G}_{N,t}$, that is, $\mathcal{G}_{N,t}$ is a proper ideal in $\mathcal{A}_{N,t}$;*

4. *$\mathcal{G}_{N,t} \cap \mathcal{G}^*_{N,t} = \emptyset$;*

5. *$\mathcal{G}_{N,t}$ is a maximal proper ideal in $\mathcal{A}_{N,t}$;*

6. *$\mathcal{G}^*_{N,t}$ is an ultrafilter in $\mathcal{A}_{N,t}$;*

7. *$\mathcal{G}^*_{N,t} = \mathcal{G}^+_{N,t}$;*

8. *$t{\downarrow} \notin I_{N,t}$, that is, $I_{N,t}$ is a proper ideal on $t{\downarrow}$;*

9. *$I^*_{N,t} \subseteq I^+_{N,t}$;*

10. *For all $A, B \in N$ with $A \cap t{\downarrow} = B \cap t{\downarrow}$, we have $t \in A \iff t \in B$ (even if $\pi_{N,t}$ is not injective);*

11. *For all $B \in N$, we have*
$$t \in B \iff B \cap t{\downarrow} \in \mathcal{G}^+_{N,t}.$$

Proof.

(1) $\implies$ (2) Clear.

$\neg$**(1) $\implies$ $\neg$(2)** If B witnesses that t is not N-eligible, then let $A = B \cap T$. Since $T \in N$, by elementarity of N we have $A \in N$, violating (2).

(1) $\iff$ (3) From the definition of $\mathcal{G}_{N,t}$.

(3) $\iff$ (4) These are always equivalent for any ideal.

(3) $\iff$ (5) From the dichotomy given by Lemma 72.

(5) $\iff$ (6) $\iff$ (7) These are always equivalent for any ideal.

(3) $\iff$ (8) From the last sentence of Lemma 74.

(8) $\iff$ (9) These are always equivalent for any ideal.

$\neg$**(10) $\implies$ $\neg$(1)** If there were $A, B \in N$ with $A \cap t{\downarrow} = B \cap t{\downarrow}$ but $t \in A \setminus B$, then $B \cup (T \setminus A)$ violates (1).

(10) $\implies$ (11) The $\impliedby$ implication in (11) is always true by definition of $\mathcal{G}_{N,t}$. If some $B \in N$ violates the $\implies$ implication, then we should have $B \cap t{\downarrow} = A \cap t{\downarrow}$ for some $A \in N$ with $t \notin A$, violating (10).

(11) $\implies$ (3) Apply the $\implies$ implication of (11) to T. □

The eligibility condition has other consequences that are not, in general, equivalent to it:

Lemma 77. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$. If $t \in T$ is N-eligible, then:*

1. $t{\downarrow} \notin N$.
2. $t \notin N$.
3. $\mathrm{ht}_T(t) \notin N$.
4. *If we also have $t{\downarrow} \subseteq N$, then*

$$\mathrm{ht}_T(t) = \min \{\delta : \delta \textit{ is an ordinal and } \delta \notin N\},$$

so that in particular t must be a limit node in that case.

Proof.

1. If $t{\downarrow} \in N$, then $t{\downarrow}$ itself would violate the N-eligibility of t.
2. If $t \in N$, then by elementarity we have $t{\downarrow} \in N$, contradicting (1).
3. Define
$$B = \{s \in T : \mathrm{ht}_T(s) < \mathrm{ht}_T(t)\}.$$
If $\mathrm{ht}_T(t) \in N$ then by elementarity we should have $B \in N$. But $t{\downarrow} \subseteq B$ and $t \notin B$, violating the N-eligibility of t.

4. For any $\beta < \mathrm{ht}_T(t)$, there must be some $s <_T t$ with $\mathrm{ht}_T(s) = \beta$. But then $s \in N$ (since by assumption $t{\downarrow} \subseteq N$), so by elementarity we have $\beta = \mathrm{ht}_T(s) \in N$. Then (3) gives the desired equation.

   For any ordinal $\beta \in N$, its successor $\beta \cup \{\beta\} \in N$ by elementarity, so the smallest ordinal not in the model must be a limit ordinal. $\square$

The following corollary follows immediately from Lemma 77(4):

Corollary 78. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$. If $s, t \in T$ are two N-eligible nodes such that $s{\downarrow}, t{\downarrow} \subseteq N$, then $\mathrm{ht}_T(s) = \mathrm{ht}_T(t)$. Furthermore, if the set*

$$\{t \in T : t \text{ is } N\text{-eligible and } t{\downarrow} \subseteq N\}$$

is nonempty, then its nodes are all at the same height, and that height is the ordinal $\min\{\delta : \delta \notin N\}$.

Remark 79. *In the special case where T is a cardinal λ and $t \geq \sup(N \cap \lambda)$, we have $N \cap T \subseteq t{\downarrow}$, so that elementarity of N implies that $\pi_{N,t}$ is one-to-one, giving an isomorphism of set algebras*

$$\pi_{N,t} : \langle \mathcal{P}(\lambda) \cap N, \cup, \cap, \setminus, \emptyset, \lambda \rangle \cong \langle \mathcal{A}_{N,t}, \cup, \cap, \setminus, \emptyset, t{\downarrow} \rangle .$$

In this case, t is necessarily N-eligible, via condition (10) of Lemma 76. So provided that $\sup(N \cap \lambda) < \lambda$ *(such as when $|N| < \lambda$ for regular cardinal λ), we can always choose an N-eligible node $t = \sup(N \cap \lambda)$ in this case.*[9] *In the general case of a tree T, it not clear that every model N necessarily has an N-eligible node, so we shall have to work harder later on to show that such models and nodes exist.*

The significance of the following lemma will become apparent when we introduce *reflection points* later on.

Lemma 80. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Fix $X \in \mathcal{A}_{N,t}$, and $S \subseteq T$ such that $S \cap t{\downarrow} \in I^+_{N,t}$. Then*

$$X \in \mathcal{G}^+_{N,t} \iff X \cap S \in I^+_{N,t}.$$

Proof. (Notice that since $S \cap t{\downarrow} \in I^+_{N,t}$, we have in particular that $I^+_{N,t} \neq \emptyset$, so that t is necessarily N-eligible. However, we do not use this fact formally in the proof.)

$\Longleftarrow$ If $X \cap S \in I^+_{N,t}$ then certainly $X \in I^+_{N,t}$. Since also $X \in \mathcal{A}_{N,t}$, Lemma 74 gives $X \in \mathcal{G}^+_{N,t}$.

$\Longrightarrow$ Suppose $X \in \mathcal{G}^+_{N,t}$. Then Lemma 72 gives $X \in \mathcal{G}^*_{N,t}$, and then by Lemma 74 also $X \in I^*_{N,t}$. By hypothesis, $S \cap t{\downarrow} \in I^+_{N,t}$. The intersection of a co-ideal set and a filter set must be in the co-ideal, and of course $X \subseteq t{\downarrow}$, so we have $X \cap S \in I^+_{N,t}$, as required. $\square$

We now consider what happens to the algebraic structures on $t{\downarrow}$ when we build a new model by *fattening* an existing one, that is, by adding sets to the model:

[9] Furthermore, if $N \cap \lambda$ is downward closed and equal to an ordinal $\delta < \lambda$, then we can choose $t = \delta$, and we have $t{\downarrow} \subseteq N$, and $\pi_{N,t}$ is just the Mostowski collapsing function of $\langle \mathcal{P}(T) \cap N, \in \rangle$ onto its transitive collapse $\mathcal{A}_{N,t}$.

Lemma 81. *Suppose $M, N \prec H(\theta)$ are two elementary submodels such that $T \in M, N$, and also $N \subseteq M$, and let $t \in T$. Then we have:*

$$\mathcal{A}_{N,t} \subseteq \mathcal{A}_{M,t}$$
$$\pi_{N,t} = \pi_{M,t} \upharpoonright (\mathcal{P}(T) \cap N)$$
$$\mathcal{G}_{N,t} \subseteq \mathcal{G}_{M,t}$$
$$I_{N,t} \subseteq I_{M,t}$$
$$I^*_{N,t} \subseteq I^*_{M,t}$$
$$I^+_{N,t} \supseteq I^+_{M,t}$$

Furthermore, if t is M-eligible then t is also N-eligible, and we have

$$\mathcal{G}_{N,t} = \mathcal{G}_{M,t} \cap \mathcal{A}_{N,t} \tag{$*$}$$
$$\mathcal{G}^+_{N,t} = \mathcal{G}^+_{M,t} \cap \mathcal{A}_{N,t}$$
$$I^*_{N,t} \subseteq I^*_{M,t} \subseteq I^+_{M,t} \subseteq I^+_{N,t}$$

Proof. Mostly straight from the definitions and previous lemmas. For $(*)$: Suppose t is M-eligible, and $X \in \mathcal{G}_{M,t} \cap \mathcal{A}_{N,t}$. We must show $X \in \mathcal{G}_{N,t}$. Since $X \in \mathcal{G}_{M,t}$, we have $X = A \cap t{\downarrow}$ for some $A \in M$ with $t \notin A$. Since $X \in \mathcal{A}_{N,t}$, we have $X = B \cap t{\downarrow}$ for some $B \in N \subseteq M$. Since t is M-eligible, and $A \cap t{\downarrow} = B \cap t{\downarrow}$ where $A, B \in M$ and $t \notin A$, condition (10) of Lemma 76 gives $t \notin B$, so that $X \in \mathcal{G}_{N,t}$, as required. □

We shall want our algebraic structures defined using a model N to be κ-complete, for some fixed cardinal κ. To ensure this, we impose the condition that the model N must contain all of its subsets of size $< \kappa$, that is, we suppose $[N]^{<\kappa} \subseteq N$. What conditions does this impose on the combinatorial relationship between κ and $|N|$?

Lemma 82. *For any infinite set A and any cardinal κ, if $[A]^{<\kappa} \subseteq A$, then $|A|^{<\kappa} = |A|$.*

Lemma 83. *Let ν and κ be infinite cardinals. If $\nu^{<\kappa} = \nu$, then $\kappa \leq \mathrm{cf}(\nu)$.*

Proof. Clearly, for every $\mu < \kappa$ we have $\nu^\mu = \nu$, that is, $\{\nu^\mu : \mu < \kappa\}$ is constant. Then by [12, Theorem 6.10(d)(i)], we have $\mathrm{cf}(\nu^{<\kappa}) \geq \kappa$. □

Further consequences on the algebraic structure due to the elementarity of N are:

Lemma 84. *Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Let κ be any cardinal. If we have $[N]^{<\kappa} \subseteq N$, then:*

1. *If $\mathcal{B} \in [N]^{<\kappa}$, then $\bigcup \mathcal{B} \in N$.*
2. *The set algebra $\mathcal{P}(T) \cap N$ is κ-complete.*
3. *The ultrafilter and maximal ideal of Lemma 69 are κ-complete.*
4. *The set algebra $\mathcal{A}_{N,t}$ is κ-complete.*
5. *The ideals $\mathcal{G}_{N,t}$ and $I_{N,t}$ are κ-complete.*

6. *If δ is the smallest ordinal not in N, then* $\operatorname{cf}(\delta) \geq \kappa$.

7. *If t is N-eligible and $t\downarrow \subseteq N$, then*
$$\kappa \leq \operatorname{cf}\left(\operatorname{ht}_T(t)\right).$$

Proof. Suppose the cardinal κ satisfies $[N]^{<\kappa} \subseteq N$.

1. Suppose $\mathcal{B} \in [N]^{<\kappa}$. Since $[N]^{<\kappa} \subseteq N$, we have $\mathcal{B} \in N$. Now $N \prec H(\theta)$, so N models a sufficient fragment of ZFC, including the union axiom, so it follows that $\bigcup \mathcal{B} \in N$, as required.

2. A union of subsets of T is certainly a subset of T, and from part (1) we know that a union of fewer than κ sets from N is in N.

3. For any collection $\mathcal{D}$ of sets in the maximal ideal, we have $t \notin B$ for each $B \in \mathcal{D}$. Certainly then,
$$t \notin \bigcup_{B \in \mathcal{D}} B.$$
So if the union $\bigcup \mathcal{D}$ is in the set algebra $\mathcal{P}(T) \cap N$, then it is in the maximal ideal as well. Since $\mathcal{P}(T) \cap N$ is κ-complete by part (2), it follows that the maximal ideal is κ-complete, and so is the dual ultrafilter.

4. By Lemma 70, the set algebra $\mathcal{A}_{N,t}$ is a homomorphic image of the κ-complete set algebra $\mathcal{P}(T) \cap N$, so it is also κ-complete.

5. The ideal $\mathcal{G}_{N,t}$ is the homomorphic image of a κ-complete ideal (from part (3)) into a κ-complete set algebra $\mathcal{A}_{N,t}$, so it is κ-complete as well.

   κ-completeness of $I_{N,t}$ follows easily.

6. Let δ be the smallest ordinal not in N, and fix any cardinal $\mu < \kappa$. We must show that $\mu < \operatorname{cf}(\delta)$.

   For each ordinal $\iota < \mu$, choose some ordinal $\gamma_\iota < \delta$. Let
$$\gamma = \sup_{\iota < \mu} \gamma_\iota = \bigcup_{\iota < \mu} \gamma_\iota,$$
   and we shall show that $\gamma < \delta$.

   Since each $\gamma_\iota < \delta$, it is clear that $\gamma \leq \delta$. But also each $\gamma_\iota \in N$, so we have
$$\{\gamma_\iota : \iota < \mu\} \in [N]^\mu \subseteq [N]^{<\kappa},$$
   so it follows from part (1) that $\gamma \in N$. Since $\delta \notin N$, it follows that $\gamma < \delta$, as required.

7. This follows immediately by combining the previous part with Lemma 77(4). □

We now consider what effect our elementary submodels have on a given colouring $c : [T]^2 \to \mu$:

Lemma 85. *Suppose we have cardinals μ and κ, with $\mu < \kappa$, and a colouring $c : [T]^2 \to \mu$. Suppose also that $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and also $[N]^{<\kappa} \subseteq N$, and let $t \in T$.*

Then for any $X \subseteq t\downarrow$, we have[10]

$$X \in I^+_{N,t} \iff \exists \text{ some colour } \chi < \mu \text{ such that } X \cap c_\chi(t) \in I^+_{N,t}.$$

Proof. For any $X \subseteq t\downarrow$, we clearly have

$$X = X \cap \bigcup_{\chi<\mu} c_\chi(t) = \bigcup_{\chi<\mu} \left(X \cap c_\chi(t)\right).$$

Since the model N satisfies $[N]^{<\kappa} \subseteq N$, Lemma 84(5) tells us that $I_{N,t}$ is κ-complete. Since $\mu < \kappa$, the required result follows. □

The following lemma contains the crucial recursive construction of a homogeneous chain of length κ as a subset of an appropriate set from the co-ideal $I^+_{N,t}$:

Lemma 86 (cf. [4, Claim before Lemma 2.2], [2, Claim 2.2])**.** *Suppose we have cardinals μ and κ, a colouring $c : [T]^2 \to \mu$, and some colour $\chi < \mu$. Suppose also that $N \prec H(\theta)$ is an elementary submodel such that $T, c, \chi \in N$, and also $[N]^{<\kappa} \subseteq N$. Let $t \in T$ be a node such that $t\downarrow \subseteq N$.*

If $X \subseteq c_\chi(t)$ is such that $X \in I^+_{N,t}$, then there is a χ-homogeneous chain $Y \in [X]^\kappa$.

Proof. We shall recursively construct a χ-homogeneous chain

$$Y = \langle y_\eta \rangle_{\eta<\kappa} \subseteq X,$$

of order type κ, as follows:

Fix some ordinal $\eta < \kappa$, and suppose we have constructed χ-homogeneous

$$Y_\eta = \langle y_\iota \rangle_{\iota<\eta} \subseteq X$$

of order type η. We need to choose $y_\eta \in X$ such that $Y_\eta <_T \{y_\eta\}$ and $Y_\eta \cup \{y_\eta\}$ is χ-homogeneous.

Since $Y_\eta \subseteq X \subseteq t\downarrow \subseteq N$ and $|Y_\eta| < \kappa$, the hypothesis that $[N]^{<\kappa} \subseteq N$ gives us $Y_\eta \in N$. Define

$$Z = \{s \in T : (\forall y_\iota \in Y_\eta) \left[y_\iota <_T s \text{ and } c\{y_\iota, s\} = \chi\right]\}.$$

Since Z is defined from parameters T, Y_η, c, and χ that are all in N, it follows by elementarity of N that $Z \in N$, so that $Z \cap t\downarrow \in \mathcal{A}_{N,t}$.

Since $Y_\eta \subseteq X \subseteq c_\chi(t)$, it follows from the definition of Z that $t \in Z$. But then we have $Z \cap t\downarrow \in \mathcal{G}^*_{N,t} \subseteq I^*_{N,t}$. By assumption we have $X \in I^+_{N,t}$. The intersection of a filter set and a co-ideal set must be in the co-ideal, so we have $X \cap Z \in I^+_{N,t}$. In particular, this set is not empty, so we choose $y_\eta \in X \cap Z$. Because $y_\eta \in Z$, we have $Y_\eta <_T \{y_\eta\}$ and $Y_\eta \cup \{y_\eta\}$ is χ-homogeneous, as required. □

Two observations about Lemma 86 will be demonstrated in the following corollary; one about the hypotheses, and the other about the conclusion.

First, implicit in the hypothesis $X \in I^+_{N,t}$ of Lemmas 85 and 86 is the fact that t is N-eligible. Ultimately, it will be the existence of N-eligible nodes that will help us find suitable sets $X \in I^+_{N,t}$ to which we can apply Lemma 86.

[10] Recall the definition of the notation $c_\chi(t)$ in chapter 2.

Second, the conclusion of Lemma 86 actually gives us a χ-homogeneous chain $Y \cup \{t\}$ of order-type $\kappa + 1$. This will be useful when we prove Theorem 53 for regular cardinals in section 4.8.

Corollary 87. *Suppose we have cardinals μ and κ, with $\mu < \kappa$, and a colouring $c : [T]^2 \to \mu$. Suppose also that $N \prec H(\theta)$ is an elementary submodel such that $T, c \in N$, and also $[N]^{<\kappa} \subseteq N$. Suppose $t \in T$ is N-eligible and $t{\downarrow} \subseteq N$. Then there is a chain $Y \in [t{\downarrow}]^\kappa$ such that $Y \cup \{t\}$ is homogeneous for c.*

Proof. Since t is N-eligible, we have by Lemma 76 that $I_{N,t}$ is a proper ideal on $t{\downarrow}$, so that $t{\downarrow} \in I^+_{N,t}$. Applying Lemma 85 to $t{\downarrow}$ itself, we fix a colour $\chi < \mu$ such that $c_\chi(t) \in I^+_{N,t}$.

Claim 87.1. *We have $\chi \in N$.*

Proof. Since $\chi < \mu < \kappa \leq \mathrm{cf}(\mathrm{ht}_T(t)) \leq \mathrm{ht}_T(t)$ (using Lemma 84(7)), there must be some $s <_T t$ with $\mathrm{ht}_T(s) = \chi$. But then $s \in N$ (since by assumption $t{\downarrow} \subseteq N$), so by elementarity we have $\chi = \mathrm{ht}_T(s) \in N$. □

We now apply Lemma 86 to $c_\chi(t)$ itself, to obtain a χ-homogeneous chain $Y \in [c_\chi(t)]^\kappa$. Then $Y \cup \{t\}$ is a χ-homogeneous chain of order type $\kappa + 1$, as required. □

So if we could ensure the existence of models N with some N-eligible nodes t such that $t{\downarrow} \subseteq N$, then we should be part way toward our goal of obtaining the long homogeneous chains we are looking for. Having built up an algebraic structure based on hypothetical elementary submodels $N \prec H(\theta)$, we should like to know: Under what circumstances can we guarantee that some models N will have some N-eligible nodes?

First, a counterexample:

Example 88. *Fix any infinite regular cardinal κ. Let T be a $(2^{<\kappa})$-special tree of height $(2^{<\kappa})^+$ (such as, for example, a special Aronszajn tree, which we know exists for $\kappa = \aleph_0$ by Theorem 21). Then by Corollary 56 we have*

$$T \not\to (\kappa + 1, \kappa)^2 .$$

Fix a colouring $c : [T]^2 \to 2$ witnessing this negative partition relation. In particular, there is no c-homogeneous chain in T of order-type $\kappa + 1$.

Claim 88.1. *If $N \prec H(\theta)$ is any elementary submodel such that $T, c \in N$ and $[N]^{<\kappa} \subseteq N$, then there is no N-eligible node $t \in T$ with $t{\downarrow} \subseteq N$, even though (provided $|N| = 2^{<\kappa}$) there are nodes $t \in T$ with $ht_T(t) = \sup(N \cap (2^{<\kappa})^+)$.*

Proof. Suppose $t \in T$ is N-eligible and $t{\downarrow} \subseteq N$. Applying Corollary 87 to the colouring c (with $\mu = 2$), we get a c-homogeneous chain of order-type $\kappa + 1$, contradicting our choice of c. □

However, $(2^{<\kappa})$-special trees are essentially the only counterexamples. We shall proceed in the next section to show how to construct collections of elementary submodels $N \prec H(\theta)$, and to show that provided T is a non-$(2^{<\kappa})$-special tree for some regular cardinal κ, we can guarantee that some of the models N satisfying $[N]^{<\kappa} \subseteq N$ will have some N-eligible nodes t such that $t{\downarrow} \subseteq N$.[11]

[11] In the special case where T is the cardinal $(2^{<\kappa})^+$ for some regular cardinal κ, and we have fixed a colouring $c : [T]^2 \to \mu$ for some cardinal $\mu < \kappa$, we can use [25, Lemma 24.28 and Claim 24.23(b)] or [37, p. 245] to fix an elementary submodel $N \prec H(\theta)$ with $T, c \in N$, such that $|N| = 2^{<\kappa}$, $[N]^{<\kappa} \subseteq N$, and $N \cap (2^{<\kappa})^+ = \delta$ for some ordinal δ with $|\delta| = 2^{<\kappa}$. Then, by Remark 79 and footnote 9 there, we can set $t = \delta$, so that t is N-eligible and $t{\downarrow} \subseteq N$. Then Corollary 87 gives us a

4.7 Very Nice Collections of Elementary Submodels

We shall generalize Kunen's definition [29, Definition III.8.14] of a *nice chain* of elementary submodels of $H(\theta)$:[12]

Definition 19. *Let λ be any regular uncountable cardinal, and let T be a tree of height λ. The collection $\langle W_t \rangle_{t \in T}$ is called a* nice collection of sets indexed by T *if:*

1. *For each $t \in T$, $|W_t| < \lambda$;*
2. *The collection is* increasing*, meaning that for $s, t \in T$ with $s <_T t$, $W_s \subseteq W_t$;*
3. *The collection is* continuous *(with respect to its indexing),*[13] *meaning that for all limit nodes $t \in T$,*

$$W_t = \bigcup_{s <_T t} W_s.$$

Suppose furthermore that $\theta \geq \lambda$ is a regular cardinal such that[14] *$T \subseteq H(\theta)$. The collection $\langle N_t \rangle_{t \in T}$ is called a* nice collection of elementary submodels of $H(\theta)$ indexed by T *if, in addition to being a nice collection of sets as above, we have:*

4. *For each $t \in T$, $N_t \prec H(\theta)$;*
5. *For each $t \in T$, $t\downarrow \subseteq N_t$;*
6. *For $s, t \in T$ with $s <_T t$, $N_s \in N_t$.*

If κ is an infinite cardinal, then we say $\langle N_t \rangle_{t \in T}$ is a κ-very nice collection of elementary submodels *if, in addition to the above conditions, we have*

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chain of order-type $\kappa + 1$, homogeneous for c. This is the proof of the balanced Erdős-Rado Theorem for regular cardinals given in [4, Section 2, Theorem 2.1].

In the slightly more general case of a tree T of height $(2^{<\kappa})^+$ such that every antichain of T has cardinality $\leq 2^{<\kappa}$ (such as a $(2^{<\kappa})^+$-Souslin tree, if one exists), we can modify the constructions of [25, Lemma 24.28] and [37, p. 245] so that, for each ordinal $\eta < \kappa$, we impose the extra condition

$$\bigcup_{B \in N_\eta} \{t \in T : t\downarrow \subseteq B \text{ and } t \notin B\} \subseteq N_{\eta+1}.$$

Then the model $N = N_\kappa$ has the property that all nodes in $T \setminus N$ are N-eligible, and in particular we have $T \cap N = \{t \in T : \mathrm{ht}_T(t) < \delta\}$, where $\delta = N \cap (2^{<\kappa})^+$. We can then choose any $t \in T$ of height δ, and as before, Corollary 87 gives us a chain of order-type $\kappa + 1$, homogeneous for c, proving the case of Theorem 53 for the tree T.

However, in the general case of an arbitrary non-$(2^{<\kappa})$-special tree, this method is insufficient. We shall need the full strength of the construction in section 4.7 in order to find models N with N-eligible nodes t such that $t\downarrow \subseteq N$, before we can return to proving Theorem 53 (for regular cardinals κ) in section 4.8.

Furthermore, our ultimate goal is to prove the Main Theorem 29, and for this we shall need many models with eligible nodes, requiring the full strength of the subsequent constructions even in the simpler cases mentioned above.

[12] Kunen's *nice chains* are the special case of our *nice collections of elementary submodels* where $T = \lambda = \omega_1$, except that we require $N_\emptyset$ to be an elementary submodel, rather than Kunen's $N_0 = \emptyset$.

[13] The continuity condition is consistent with the topological notion of continuity if we define the appropriate topologies: The topology on $\mathcal{P}(W)$, where $W = \bigcup_{t \in T} W_t$, is the product topology on $\{0,1\}^W$, that is, the topology of pointwise convergence on the set of characteristic functions on W, where of course $\{0,1\}$ has the discrete topology.

The topology on T is the *tree topology*, defined earlier in footnote 6 on page 24.

[14] We could simplify the requirements on θ in this definition and in the subsequent lemmas by requiring the stronger condition $T \in H(\theta)$. This implies both of the required conditions $T \subseteq H(\theta)$ and $\theta \geq \lambda$, as well as the extra condition $\theta > |T|$, but this seems to be unnecessary.

7. *For $s, t \in T$ with $s <_T t$, $[N_s]^{<\kappa} \subseteq N_t$.*

If $\langle M_t \rangle_{t\in T}$ and $\langle N_t \rangle_{t\in T}$ are two nice collections of sets, then we say that $\langle N_t \rangle_{t\in T}$ is a fattening *of $\langle M_t \rangle_{t\in T}$ if for all $t \in T$ we have $M_t \subseteq N_t$.*

Notice that all nice collections of elementary submodels are $\aleph_0$-very nice collections, since any elementary submodel of $H(\theta)$ contains all of its finite subsets.

What condition on the combinatorial relationship between κ and λ is necessary for the existence of a κ-very nice collection of elementary submodels indexed by a tree T of height λ?

Lemma 89. *Suppose λ is any regular uncountable cardinal, T is a tree of height λ, and $\theta \geq \lambda$ is a regular cardinal such that $T \subseteq H(\theta)$. Suppose κ is any infinite cardinal. If there exists a κ-very nice collection of elementary submodels of $H(\theta)$ indexed by T, then we must have*

$$(\forall \text{ cardinals } \nu < \lambda) \left[\nu^{<\kappa} < \lambda\right]. \tag{$**$}$$

Proof. Let $\langle N_t \rangle_{t\in T}$ be a κ-very nice collection of elementary submodels of $H(\theta)$, and fix any cardinal $\nu < \lambda$. Since T has height λ, we can choose some $s, t \in T$ with $\mathrm{ht}_T(s) = \nu$ and $s <_T t$. Then $s{\downarrow} \subseteq N_s$, so that $|N_s| \geq |s{\downarrow}| = \nu$. Since the collection is κ-very nice, we must have $[N_s]^{<\kappa} \subseteq N_t$, so that

$$|N_t| \geq \left|[N_s]^{<\kappa}\right| = |N_s|^{<\kappa} \geq \nu^{<\kappa}.$$

But we must also have $|N_t| < \lambda$, giving the requirement $\nu^{<\kappa} < \lambda$. □

In the intended applications, the height λ of our tree will be a successor cardinal ν^+. In that case, condition $(**)$ becomes simply $\nu^{<\kappa} = \nu$, from which we obtain the following chain of equations and inequalities (using Lemma 83 for one of them), which we shall refer to when necessary:

$$\kappa \leq \mathrm{cf}(\nu) \leq \nu = \nu^{<\kappa} < \nu^+ = \mathrm{ht}(T)$$

In general, it turns out that the necessary condition $(**)$ is also sufficient, as the following lemma shows (particularly, part (3)):

Lemma 90. *Suppose λ is any regular uncountable cardinal, T is a tree of height λ, and $\theta \geq \lambda$ is a regular cardinal such that $T \subseteq H(\theta)$. Fix $X \subseteq H(\theta)$ with $|X| < \lambda$. Then:*

1. *There is a nice collection $\langle N_t \rangle_{t\in T}$ of elementary submodels of $H(\theta)$ such that $X \subseteq N_\emptyset$ (and therefore $X \subseteq N_t$ for every $t \in T$).*

2. *Given any nice collection $\langle M_t \rangle_{t\in T}$ of elementary submodels of $H(\theta)$, we can fatten the collection to include X, that is, we can construct another nice collection $\langle N_t \rangle_{t\in T}$ of elementary submodels of $H(\theta)$, that is a fattening of $\langle M_t \rangle_{t\in T}$, such that $X \subseteq N_\emptyset$.*

3. *If κ is an infinite cardinal such that for all cardinals $\nu < \lambda$ we have $\nu^{<\kappa} < \lambda$, then the nice collections we construct in parts (1) and (2) can be κ-very nice collections.*

Proof. We construct the nice collection recursively. The Downward Löwenheim-Skolem-Tarski Theorem guarantees the existence of elementary submodels of arbitrary infinite cardinality, and a version of it given in [29, Theorem I.15.10], [25, Corollary 24.13], [9, Theorem 1.1], and [37, Theorem 2] says that the submodel can even be guaranteed to contain any number of specified items, up to the cardinality of the desired submodel. This is our main tool for the construction, which proceeds as follows:

For $\emptyset$

1. By the Downward Löwenheim-Skolem-Tarski Theorem version just mentioned, we can choose $N_\emptyset \prec H(\theta)$ such that $X \subseteq N_\emptyset$, with

$$|N_\emptyset| = \max\{|X|, \aleph_0\} < \lambda,$$

satisfying the required properties.

2. If we are fattening an already-existing collection, there is no difficulty in ensuring as well that $M_\emptyset \subseteq N_\emptyset$. In this case we should have

$$|N_\emptyset| = \max\{|X|, |M_\emptyset|\} < \lambda.$$

3. There is no additional requirement on $N_\emptyset$ in a κ-very nice collection.

For successor nodes

1. Fix $s \in T$, and assume that we have already constructed $N_s \prec H(\theta)$ satisfying the required properties, and suppose that $t \in T$ is an immediate successor of s. Again, by the Downward Löwenheim-Skolem-Tarski Theorem, we can choose $N_t \prec H(\theta)$ such that

$$N_s \cup \{N_s\} \cup t{\downarrow} \subseteq N_t,$$

with $|N_t| = |N_s| < \lambda$. The required properties are easy to verify.

2. If we are fattening an already-existing collection, then again there is no difficulty in ensuring as well that $M_t \subseteq N_t$. In this case we should have

$$|N_t| = \max\{|N_s|, |M_t|\} < \lambda.$$

3. Since $|N_s| < \lambda$, the extra hypothesis in this part gives us

$$\left|[N_s]^{<\kappa}\right| \leq |N_s|^{<\kappa} < \lambda,$$

so that we can choose N_t such that

$$[N_s]^{<\kappa} \subseteq N_t,$$

while still having $|N_t| < \lambda$.

For limit nodes Fix limit node $t \in T$, and assume that we have already constructed the chain $\langle N_s : s <_T t\rangle$ satisfying the required properties. Define

$$N_t = \bigcup_{s <_T t} N_s.$$

As the union of an increasing chain of elementary submodels is an elementary submodel ([25, Lemma 24.5], [9, Corollary 1.3], and [37, top of p. 245]), we have $N_t \prec H(\theta)$. Since λ is a regular cardinal, and $\mathrm{ht}_T(t) < \lambda$ (so that $|t{\downarrow}| < \lambda$), and each $|N_s| < \lambda$, it is clear that $|N_t| < \lambda$. The remaining properties are easy to verify. □

Given any tree T of height λ, any large enough θ, and any cardinal κ satisfying condition $(**)$, we can use Lemma 90 to construct a κ-very nice collection of elementary submodels $\langle N_t \rangle_{t\in T}$ of $H(\theta)$, such that $N_\emptyset$ contains any relevant sets, and in particular we can ensure that $T \in N_\emptyset$. We can then use any node $t \in T$ and its associated model N_t to build the algebraic structures defined in section 4.6, including the ideal $I_{N_t,t}$ on $t{\downarrow}$. By definition of our nice collections, we always have $N_t \prec H(\theta)$ and $t{\downarrow} \subseteq N_t$, but in order to get the most value from these structures, we shall need to find nodes and models with two extra features: eligibility and κ-completeness.

First, κ-completeness of the required algebraic structures (and, in particular, the recursive construction of Lemma 86) depends on the additional condition $[N]^{<\kappa} \subseteq N$ introduced before Lemma 82. For any $s <_T t$ in T, we have $[N_s]^{<\kappa} \subseteq N_t$. But we want to know: Which nodes $t \in T$ can we choose so that the model N_t satisfies the stronger condition $[N_t]^{<\kappa} \subseteq N_t$?

Lemma 91. *Suppose λ is any regular uncountable cardinal, T is a tree of height λ, and $\theta \geq \lambda$ is a regular cardinal such that $T \subseteq H(\theta)$. Suppose κ is any infinite cardinal, and $\langle N_t \rangle_{t\in T}$ is a κ-very nice collection of elementary submodels of $H(\theta)$. Then for every $t \in T$, we have:*

1. *If $\mathrm{cf}(\mathrm{ht}_T(t)) \geq \kappa$ then $[N_t]^{<\kappa} \subseteq N_t$.*
2. *If t is N_t-eligible, then*[15]
$$\mathrm{cf}\left(\mathrm{ht}_T(t)\right) \geq \kappa \iff [N_t]^{<\kappa} \subseteq N_t.$$

Proof.

1. Fix $t \in T$ such that $\mathrm{cf}(\mathrm{ht}_T(t)) \geq \kappa$. Fix a cardinal $\mu < \kappa$, and some collection
$$\mathcal{C} = \langle A_\iota \rangle_{\iota<\mu} \in [N_t]^\mu .$$
For each ordinal $\iota < \mu$, we have $A_\iota \in N_t$. Since $\mathrm{cf}(\mathrm{ht}_T(t)) \geq \kappa$, t must be a limit node, so since the collection of models is continuous, we have $A_\iota \in N_{s_\iota}$ for some $s_\iota <_T t$. Then define
$$s = \sup_{\iota<\mu} s_\iota,$$
where the sup is taken along the chain $t{\downarrow} \cup \{t\}$. Since each $s_\iota <_T t$ and $\mu < \kappa \leq \mathrm{cf}(\mathrm{ht}_T(t))$, we have $s <_T t$. We then have, since the collection is κ-very nice,
$$\mathcal{C} \in [N_s]^\mu \subseteq [N_s]^{<\kappa} \subseteq N_t,$$
as required.
2. This is simply a combination of the previous part with Lemma 84(7). □

Provided we start with a non-special tree, this guarantees a large supply of κ-complete models:

[15] This part is not actually used.

Lemma 92. *Suppose ν is any infinite cardinal, T is a non-ν-special tree (necessarily of height ν^+), and $\theta > \nu$ is a regular cardinal such that $T \subseteq H(\theta)$. Suppose κ is an infinite cardinal, and $\langle N_t \rangle_{t\in T}$ is a κ-very nice collection of elementary submodels of $H(\theta)$. Then the set*

$$\left\{t \in T : [N_t]^{<\kappa} \subseteq N_t\right\}$$

is a stationary subtree of T.

Proof. Since T is a non-ν-special tree, Theorem 52 gives[16]

$$T \restriction S^{\nu^+}_{\mathrm{cf}(\nu)} = \{t \in T : \mathrm{cf}(\mathrm{ht}_T(t)) = \mathrm{cf}(\nu)\} \notin NS^T_\nu.$$

Since there exists a κ-very nice collection of elementary submodels indexed by T, condition $(**)$ must be satisfied, so that $\nu^{<\kappa} = \nu$. Then Lemma 83 gives $\kappa \leq \mathrm{cf}(\nu)$. For any $t \in T \restriction S^{\nu^+}_{\mathrm{cf}(\nu)}$, we have $\mathrm{cf}(\mathrm{ht}_T(t)) = \mathrm{cf}(\nu) \geq \kappa$, so by Lemma 91(1) we have $[N_t]^{<\kappa} \subseteq N_t$. It follows that

$$T \restriction S^{\nu^+}_{\mathrm{cf}(\nu)} \subseteq \left\{t \in T : [N_t]^{<\kappa} \subseteq N_t\right\},$$

so that this last set is also a stationary subtree of T, as required. □

Next, recall the earlier eligibility condition for nodes and models: Given a nice collection of sets $\langle W_t \rangle_{t\in T}$, the node $t \in T$ is W_t-eligible if

$$\nexists B \in W_t [t\downarrow \subseteq B \text{ and } t \notin B].$$

We should like to know that not too many nodes t are W_t-*ineligible*.

Lemma 93. *Suppose ν is any infinite cardinal, and let T be a tree of height ν^+. Suppose $\langle W_t \rangle_{t\in T}$ is a nice collection of sets. Then*

$$\{t \in T : t \textit{ is not } W_t\textit{-eligible } \} \in NS^T_\nu.$$

Proof. For any fixed set B, the set $\{t \in T : t\downarrow \subseteq B \text{ and } t \notin B\}$ is an antichain. For any $s \in T$, we have $|W_s| \leq \nu$, so it follows that

$$\bigcup_{B\in W_s} \{t \in T : t\downarrow \subseteq B \text{ and } t \notin B\}$$

is a union of $\leq \nu$ antichains, that is, it is a ν-special subtree.

Since the set of successor nodes is always a nonstationary subtree by Lemma 51(3), we can consider only limit nodes. Suppose t is a limit node. Then by continuity of the nice collection $\langle W_t \rangle_{t\in T}$, if $B \in W_t$

[16] In the definition of this stationary subtree, we can replace $= \mathrm{cf}(\nu)$ with $\geq \mathrm{cf}(\nu)$, if desired, or even with $\geq \kappa$ (since $\mathrm{cf}(\nu) \geq \kappa$). In the special case where T is the cardinal ν^+, the more general textbook theorem applies (see our comment before Theorem 52), so that if κ is regular, we can alternatively use $= \kappa$, as is done in the definition of S_0 given in [4, p. 5].

then $B \in W_s$ for some $s <_T t$. So

$$\begin{aligned}
&\{\text{limit nodes } t \text{ that are not } W_t\text{-eligible }\}\\
&= \{\text{limit } t \in T : \exists s <_T t \exists B \in W_s\, [t\!\downarrow\, \subseteq B \text{ and } t \notin B]\}\\
&= \mathop{\nabla}_{s\in T} \{\text{limit } t \in T : \exists B \in W_s\, [t\!\downarrow\, \subseteq B \text{ and } t \notin B]\}\\
&= \mathop{\nabla}_{s\in T} \bigcup_{B\in W_s} \{\text{limit } t \in T : [t\!\downarrow\, \subseteq B \text{ and } t \notin B]\} \in NS_\nu^T,
\end{aligned}$$

and it follows that the set of nodes t such that t is not W_t-eligible is in NS_ν^T, as required. □

Combining the last two lemmas, we are guaranteed a large supply of nodes and models that satisfy both the κ-completeness and eligibility requirements, provided we start with a non-special tree and that condition $(**)$ holds:

Corollary 94. *Suppose ν is any infinite cardinal, T is a non-ν-special tree (necessarily of height ν^+), and $\theta > \nu$ is a regular cardinal such that $T \subseteq H(\theta)$. Suppose κ is an infinite cardinal, and $\langle N_t\rangle_{t\in T}$ is a κ-very nice collection of elementary submodels of $H(\theta)$. Then the set*

$$\left\{t \in T : t \text{ is } N_t\text{-eligible and } [N_t]^{<\kappa} \subseteq N_t\right\}$$

is a stationary subtree of T.

Proof. From Lemma 92, the set

$$\left\{t \in T : [N_t]^{<\kappa} \subseteq N_t\right\}$$

is a stationary subtree of T. By Lemma 93, we have

$$\{t \in T : t \text{ is not } N_t\text{-eligible }\} \in NS_\nu^T.$$

Our desired set is obtained by subtracting a nonstationary subtree from a stationary subtree, so it must be stationary, as required. □

4.8 Erdős-Rado Theorem for Trees

To demonstrate the power of the tools we have developed in the previous two sections, we now (similarly to [4, Section 2]) divert our attention from the Main Theorem to show how the machinery we have developed allows us to prove Theorem 53 for the case where κ is a regular cardinal:

Proof of Theorem 53 for regular κ. As in the hypotheses, fix an infinite regular cardinal κ, and a non-$(2^{<\kappa})$-special tree T (necessarily of height $(2^{<\kappa})^+$), and let $c : [T]^2 \to \mu$ be a colouring, where $\mu < \kappa$. We are looking for a chain of order type $\kappa + 1$, homogeneous for c.

Let θ be any regular cardinal large enough so that $T \in H(\theta)$. Using Theorem 63, we have $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$. Then we use Lemma 90 (parts (1) and (3)) to fix a κ-very nice collection $\langle N_t\rangle_{t\in T}$ of elementary submodels of $H(\theta)$ such that $T, c \in N_\emptyset$.

Since T is a non-$(2^{<\kappa})$-special tree, Corollary 94 tells us that the set

$$\left\{t \in T : t \text{ is } N_t\text{-eligible and } [N_t]^{<\kappa} \subseteq N_t\right\}$$

is a stationary subtree of T, so we can choose some node $t \in T$ such that t is N_t-eligible and $[N_t]^{<\kappa} \subseteq N_t$. Fix such a node t. Since N_t was taken from a nice collection of elementary submodels, we have $t{\downarrow} \subseteq N_t$. Then Corollary 87 gives us a chain of order-type $\kappa + 1$ homogeneous for c, as required. □

4.9 Proof of the Main Theorem

In this section, we shall prove the Main Theorem, Theorem 29.

As in the hypotheses of the Main Theorem 29, fix an infinite regular cardinal κ. Let $\nu = 2^{<\kappa}$. Fix a non-ν-special tree T (necessarily of height ν^+), a natural number k, and a colouring

$$c : [T]^2 \to k.$$

Since κ is regular, Theorem 63 gives us $\nu^{<\kappa} = \nu$, a fact that will be essential in the proof.

Let θ be any regular cardinal large enough so that $T \in H(\theta)$. Using Lemma 90 (parts (1) and (3)) and the fact that $\nu^{<\kappa} = \nu$, fix a κ-very nice collection $\langle N_t \rangle_{t \in T}$ of elementary submodels of $H(\theta)$ such that $T, c \in N_\emptyset$.

The proof of Theorem 53 in section 4.8 relied on Corollary 87, where we were able to obtain a homogeneous set of order type $\kappa + 1$ relatively easily using the co-ideal $I^+_{N,t}$ constructed from a single elementary submodel N. However, to obtain a homogeneous set of order type $\kappa + \xi$, where $\xi > 1$, we need to do some more work. In particular, it will not be so easy to determine, initially, which $i < k$ will be the colour of the required homogeneous set, so we must devise a technique for describing sets that simultaneously include homogeneous subsets for several colours. For this, we shall need some more machinery.

Recall that in section 4.6 we used nodes $t \in T$ and models N to create algebraic structures on $t{\downarrow}$, including the ideal $I_{N,t}$. Now that we have fixed a nice collection of elementary submodels indexed by T, we shall generally allow the node t to determine the model N_t and therefore the corresponding structures on $t{\downarrow}$. We shall therefore simplify our notation as follows:

Definition 20. *We define, for each $t \in T$:*

$$\pi_t = \pi_{N_t,t};$$
$$\mathcal{A}_t = \mathcal{A}_{N_t,t};$$
$$\mathcal{G}_t = \mathcal{G}_{N_t,t};$$
$$I_t = I_{N_t,t}.$$

Furthermore, we shall say that t is eligible *if it is N_t-eligible.*

Lemma 95. *Fix $r \in T$ and $B \in N_r$. Then:*

1. *For all $s \geq_T r$, we have $B \in N_s$ and $B \cap s{\downarrow} \in \mathcal{A}_s$.*

2. For all eligible nodes $s \geq_T r$, we have

$$s \in B \iff B \cap s{\downarrow} \in \mathcal{G}_s^+ \iff B \cap s{\downarrow} \in I_s^+.$$

Proof.

1. This result follows from the fact that the nice collection of models is increasing, as well as the definition of $\mathcal{A}_s$.

2. Since $B \in N_s$ from part (1), this result follows from Lemmas 76 and 74. □

Definition 21. *Let $S \subseteq T$ be any subtree, and suppose $t \in T$. If $S \cap t{\downarrow} \in I_t^+$, then t is called a* reflection point *of S.*

Some easy facts about reflection points:

Lemma 96.

1. *If $t \in T$ is a reflection point of some subtree $S \subseteq T$, then t is eligible.*

2. *If $t \in T$ is a reflection point of S, then t is a limit point of S.*

3. *If $R \subseteq S \subseteq T$ and $t \in T$ is a reflection point of R, then t is a reflection point of S.*

Proof.

1. Since $S \cap t{\downarrow} \in I_t^+$, we have in particular that $I_t^+ \neq \emptyset$, which is equivalent by Lemma 76 to t being eligible.

2. Since t is eligible by part (1), and also $t{\downarrow} \subseteq N_t$, Lemma 77(4) tells us that t must be a limit node. By Lemma 75(3), since $S \cap t{\downarrow} \in I_t^+$, $S \cap t{\downarrow}$ must be cofinal in $t{\downarrow}$. It follows that t must be a limit point of $S \cap t{\downarrow}$, and therefore also of S.

3. I_t^+ is a co-ideal and therefore closed under supersets. □

We want to be able to know when some eligible $t \in T$ is a reflection point of some subtree $S \subseteq T$. Is it enough to assume that $t \in S$? If $S \in N_t$ and $t \in S$ is eligible, then we have $S \cap t{\downarrow} \in \mathcal{G}_t^* = G_t^+ \subseteq I_t^+$ by Lemma 76, so that t is a reflection point of S. Furthermore, if $S \in N_t$ for some $t \in T$, the combination of Lemma 95(2) and Lemma 96(1) tells us precisely which $u \in t{\uparrow}$ are reflection points of S, namely those eligible $u \in t{\uparrow}$ such that $u \in S$. But what if $S \notin N_t$? Then we can't guarantee that every eligible $t \in S$ is a reflection point of S, but we can get close. The following lemma will be applied several times throughout the proof of the Main Theorem:

Lemma 97 (cf. [4, Lemma 3.2])**.** *For any $S \subseteq T$, we have*

$$\{t \in S : S \cap t{\downarrow} \in I_t\} \in NS_\nu^T.$$

In the case where S itself is a nonstationary subtree, Lemma 97 is trivially true. But then the result is also useless, as we do not obtain any reflection points. The significance of the lemma is when S is stationary in T. In that case, the lemma tells us that "almost all" points of S are reflection points: the set of points of S that are *not* reflection points is a nonstationary subtree of T. In [4, Lemma 3.2] (dealing with the special case where the tree is a cardinal), the lemma is stated for stationary sets S, and the conclusion is worded differently, but the fact that S is stationary is not actually used at all in the proof.

Proof of Lemma 97. Recall that the problem was that S is not necessarily in any of the models N_t already defined. We therefore *fatten* the models to include $\{S\}$. That is, we use Lemma 90(2) to construct another nice collection $\langle M_t \rangle_{t\in T}$ of elementary submodels of $H(\theta)$ that is a fattening of the collection $\langle N_t \rangle_{t\in T}$ (meaning that for all $t \in T$ we have $N_t \subseteq M_t$), and such that $S \in M_\emptyset$ (so that $S \in M_t$ for all $t \in T$). The idea is that while initially there may be some eligible nodes in S that are not reflection points of S, by fattening the models to contain S all of those points will become ineligible with respect to the new collection of models, showing that there cannot be too many of them.

More precisely: Fix any $t \in T$. We have $S \in M_t$. If t is M_t-eligible, then we have (using Lemmas 76 and 74)

$$t \in S \iff S \cap t{\downarrow} \in \mathcal{G}^*_{M_t,t} = \mathcal{G}^+_{M_t,t} \iff S \cap t{\downarrow} \in I^+_{M_t,t}.$$

Since $N_t \subseteq M_t$, we apply Lemma 81 to get

$$I^+_{M_t,t} \subseteq I^+_{N_t,t} = I^+_t.$$

It follows that if $t \in S$ is M_t-eligible, then $S \cap t{\downarrow} \in I^+_t$, so that t is a reflection point of S. Equivalently, if $t \in S$ satisfies $S \cap t{\downarrow} \in I_t$, then t must not be M_t-eligible.

Applying Lemma 93 to the nice collection $\langle M_t \rangle_{t\in T}$, we then have

$$\{t \in S : S \cap t{\downarrow} \in I_t\} \subseteq \{t \in T : t \text{ is not } M_t\text{-eligible } \} \in NS^T_\nu.$$

which is the required result. □

Definition 22. *We define subtrees $S_n \subseteq T$, for $n \leq \omega$, by recursion on n, as follows:*[17]

First, define

$$S_0 = \left\{t \in T : t \textit{ is eligible and } [N_t]^{<\kappa} \subseteq N_t\right\}.$$

Then, for every $n < \omega$, define

$$S_{n+1} = \left\{t \in S_n : S_n \cap t{\downarrow} \in I^+_t\right\}.$$

Finally, define

$$S_\omega = \bigcap_{n<\omega} S_n.$$

Lemma 98. *The sequence $\langle S_n \rangle_{n\leq\omega}$ satisfies the following properties:*

1. *The sequence is decreasing, that is,*

$$T \supseteq S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq S_\omega.$$

2. *For all $n < m \leq \omega$, each $t \in S_m$ is a reflection point of S_n, and therefore also a limit point of S_n.*

[17] It may be possible to omit the requirement "t is eligible" from the definition of S_0. Ultimately, this should not be a problem, as the subsequent sets S_n (for $n > 0$) will consist only of reflection points (by Lemma 98(2)), which are eligible by Lemma 96(1). However, some of the subsequent lemmas will have to be qualified, such as Lemma 99 and Corollary 103.

3. *For all $n < m \leq \omega$, the set $S_n \setminus S_m$ is a nonstationary subtree of T.*

4. *For all $n \leq \omega$, S_n is a stationary subtree of T.*

Proof.

1. Straight from the definition.

2. For every $n < m \leq \omega$ we have $S_m \subseteq S_{n+1}$ from (1), and S_{n+1} consists only of reflection points of S_n by definition. By Lemma 96(2), any reflection point of S_n must also be a limit point of S_n.

3. For each $j < \omega$, we have
$$S_j \setminus S_{j+1} = \{t \in S_j : S_j \cap t{\downarrow} \in I_t\},$$
and this subtree is nonstationary by Lemma 97. We then have, for any $n < m \leq \omega$,
$$S_n \setminus S_m = \bigcup_{n \leq j < m} (S_j \setminus S_{j+1}),$$
so this subtree is nonstationary, as it is the union of at most countably many nonstationary subtrees.

4. Since T is a non-ν-special tree, the fact that S_0 is stationary is Corollary 94.

   For $0 < n \leq \omega$, we know from (3) that $S_0 \setminus S_n$ is nonstationary, so it follows that S_n is stationary. □

Lemma 99.

1. *For every $r \in T$ and $B \in N_r$, we have*
$$S_0 \cap (\{r\} \cup r{\uparrow}) \cap B = \left\{s \in S_0 \cap (\{r\} \cup r{\uparrow}) : B \cap s{\downarrow} \in \mathcal{G}_s^+\right\}.$$

2. *For every $r \leq_T t$ in T and $B \in N_r$, we have*
$$S_0 \cap (t{\downarrow} \setminus r{\downarrow}) \cap B = \left\{s \in S_0 \cap (t{\downarrow} \setminus r{\downarrow}) : B \cap s{\downarrow} \in \mathcal{G}_s^+\right\}.$$

Proof.

1. This follows from Lemma 95(2), using the fact that nodes in S_0 are eligible.

2. This follows from part (1), since (for $r \leq_T t$)
$$t{\downarrow} \setminus r{\downarrow} \subseteq \{r\} \cup r{\uparrow}.$$
□

We shall now define ideals $I(t, \sigma)$ and $J(t, \sigma)$ on $t{\downarrow}$, for certain nodes $t \in T$ and finite sequences of colours $\sigma \in k^{<\omega}$. We continue to follow the convention as explained in chapter 2, that properness is *not* required for a collection of sets to be called an ideal (or a filter). Some of the ideals we are about to define may not be proper.

Though we define the ideals $I(t, \sigma)$ and $J(t, \sigma)$, our intention will be to focus on the corresponding co-ideals, just as we said earlier regarding the co-ideals I_t^+ corresponding to the ideals I_t. As we shall see (Lemma 107), for a set to be in some co-ideal $I^+(t, \sigma)$ implies that it will include homogeneous sets of size κ for every colour in the sequence σ. This gives us the flexibility to choose later which colour in

σ will be used when we combine portions of such sets to get a set of order type $\kappa + \xi$, homogeneous for the colouring c. When reading the definitions, it will help the reader's intuition to think of the co-ideals rather than the ideals.

Definition 23. *We shall define ideals $J(t,\sigma)$ and $I(t,\sigma)$ jointly by recursion on the length of the sequence σ. The collection $J(t,\sigma)$ will be defined for all $\sigma \in k^{<\omega}$ but only when $t \in S_{|\sigma|}$, while the collection $I(t,\sigma)$ will be defined only for nonempty sequences σ but for all $t \in S_{|\sigma|-1}$.*

(When $\sigma \in k^n$ we say $|\sigma| = n$.)

- *Begin with the empty sequence, $\sigma = \langle\rangle$. For $t \in S_0$, we define*

$$J(t, \langle\rangle) = I_t.$$

- *Fix $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|}$, and assume we have defined $J(t,\sigma)$. Then, for each colour $i < k$, we define $I(t, \sigma^\frown \langle i\rangle) \subseteq \mathcal{P}(t\downarrow)$ by setting, for $X \subseteq t\downarrow$,*

$$X \in I(t, \sigma^\frown \langle i\rangle) \iff X \cap c_i(t) \in J(t,\sigma).$$

- *Fix $\sigma \in k^{<\omega}$ with $\sigma \neq \emptyset$, and assume we have defined $I(s,\sigma)$ for all $s \in S_{|\sigma|-1}$. Fix $t \in S_{|\sigma|}$. We define $J(t,\sigma) \subseteq \mathcal{P}(t\downarrow)$ by setting, for $X \subseteq t\downarrow$,*

$$X \in J(t,\sigma) \iff \left\{ s \in S_{|\sigma|-1} \cap t\downarrow : X \cap s\downarrow \in I^+(s,\sigma) \right\} \in I_t.$$

We have introduced the intermediary $J(t,\sigma)$ as an intuitive aid to understand the recursive construction and subsequent inductive proofs. As can be seen by examining the definition, each I-ideal is defined from a J-ideal by extending the sequence of colours σ, without changing the node t; but each J-ideal is defined by considering I-ideals from nodes lower down in the tree, without changing the sequence of colours.

In fact the collections $I(t,\sigma)$ can be described explicitly without the intermediary $J(t,\sigma)$ (as in [4]), by saying, for $\sigma \neq \emptyset$,

$$X \in I(t, \sigma^\frown \langle i\rangle) \iff \left\{ s \in S_{|\sigma|-1} \cap t\downarrow : X \cap c_i(t) \cap s\downarrow \in I^+(s,\sigma) \right\} \in I_t.$$

However, we have changed the base case from [4]: We do not define $I(t,\langle\rangle)$, and by setting $J(t,\langle\rangle) = I_t$, we eliminate an unnecessary reflection step for the sequences of length 1. This way, $I(t,\langle i\rangle)$ is defined in a more intuitive way, by setting (for $X \subseteq t\downarrow$)

$$X \in I(t, \langle i\rangle) \iff X \cap c_i(t) \in I_t,$$

and this definition is valid for all $t \in S_0$, not just in S_1. Subsequent lemmas are easier to prove with this definition, and this seems to be how $I(t,\sigma)$ is intuitively understood, even in [4].

In particular, Corollary 103 will be proven much more easily, and its meaning is the intended intuitive one, which was not the case under the original definition of $I(t,\langle i\rangle)$ in [4]. Furthermore, Lemma 113 would not have been true using the original definition in [4].

We now investigate some properties of the various collections $I(t,\sigma)$ and $J(t,\sigma)$ and the relationships between them.

Lemma 100. *For each sequence σ and each relevant t, the collections $I(t,\sigma)$ and $J(t,\sigma)$ are κ-complete ideals on $t\downarrow$ (though not necessarily proper).*

Proof. Easy, by induction over the length of the sequence σ, using the fact that we are using only nodes $t \in S_0$, so that each I_t is a κ-complete ideal on $t\downarrow$ (Lemma 84(5)). □

We are going to need to take sets from co-ideals, so it would help us to get a sense of when the ideals are proper.

For a fixed t and σ, there is no particular relationship between $I(t,\sigma)$ and any $I(t,\sigma^\frown\langle i\rangle)$. In fact there is no relationship between $I(t,\sigma)$ and $J(t,\sigma)$, as the latter is defined in terms of $I(s,\sigma)$ for $s <_T t$ only. We need to explore relationships that do exist between various ideals.

Lemma 101. *For each sequence $\sigma \in k^{<\omega}$ and each $t \in S_{|\sigma|}$, we have*

$$J(t,\sigma) = \bigcap_{i<k} I(t,\sigma^\frown\langle i\rangle),$$

and equivalently,

$$J^+(t,\sigma) = \bigcup_{i<k} I^+(t,\sigma^\frown\langle i\rangle).$$

In particular, if $J(t,\sigma)$ is proper, then for at least one $i<k$, $I(t,\sigma^\frown\langle i\rangle)$ must be proper.

Proof. For $X \subseteq t\downarrow$, we have

$$\begin{aligned}
X \in J(t,\sigma) &\iff X \cap \bigcup_{i<k} c_i(t) \in J(t,\sigma) && \left(\text{since } t\downarrow = \bigcup_{i<k} c_i(t)\right)\\
&\iff \bigcup_{i<k} (X \cap c_i(t)) \in J(t,\sigma)\\
&\iff \forall i<k\,[X \cap c_i(t) \in J(t,\sigma)] && (\text{since } J(t,\sigma) \text{ is an ideal})\\
&\iff \forall i<k\,[X \in I(t,\sigma^\frown\langle i\rangle)]\\
&\iff X \in \bigcap_{i<k} I(t,\sigma^\frown\langle i\rangle)
\end{aligned}$$

□

The following special case of Lemma 101 where $\sigma = \langle\rangle$ can be thought of as a reformulation of Lemma 85 using the terminology of our new ideals $J(t,\langle\rangle)$ and $I(t,\langle i\rangle)$:

Corollary 102. *For each $t \in S_0$, we have*

$$I_t^+ = J^+(t,\langle\rangle) = \bigcup_{i<k} I^+(t,\langle i\rangle).$$

In particular, since S_0 consists only of eligible nodes, any $t \in S_0$ satisfies $t\downarrow \in I_t^+$ by Lemma 76, so applying Corollary 102 to $t\downarrow$ gives:

Corollary 103. *For each $t \in S_0$, there is some colour $i<k$ such that $I(t,\langle i\rangle)$ is proper, that is, $I^+(t,\langle i\rangle)$ is nonempty.*[18]

[18] This result is not actually used, but it corresponds to the sentence at the bottom of [4, p. 5].

Recall that each S_{n+1} consists of reflection points of S_n. This is for a good reason: If we were to allow $t \in S_{n+1}$ that was not a reflection point of S_n, then we should have $S_n \cap t\!\downarrow \in I_t$, so for all $\sigma \in k^{n+1}$ we should have $t\!\downarrow \in J(t,\sigma)$, so that $J(t,\sigma)$ could not be proper, regardless of the sequence σ. In contrast, since we allow only reflection points in $S_{|\sigma|}$ each time we lengthen σ, we obtain the following lemma:

Lemma 104.

1. *For all $n \geq 0$ and all $t \in S_n$, we have*[19]

$$\mathcal{G}_t^+ = \bigcup_{\sigma \in k^n} J^+(t,\sigma) \cap \mathcal{A}_t,$$

and equivalently,

$$\mathcal{G}_t = \bigcap_{\sigma \in k^n} J(t,\sigma) \cap \mathcal{A}_t.$$

2. *Similarly, For all $n \geq 1$ and all $t \in S_{n-1}$, we have*

$$\mathcal{G}_t^+ = \bigcup_{\sigma \in k^n} I^+(t,\sigma) \cap \mathcal{A}_t,$$

and equivalently,

$$\mathcal{G}_t = \bigcap_{\sigma \in k^n} I(t,\sigma) \cap \mathcal{A}_t.$$

Proof. We prove parts (1) and (2) jointly by induction on n.

Base case for (1) When $n = 0$, the only $\sigma \in k^0$ is the empty sequence $\langle\rangle$, and for any $t \in S_0$, $J(t,\langle\rangle)$ is defined to equal I_t, so the equality reduces to Lemma 74.

Induction step, (1) $\implies$ (2) Fix $n \geq 0$ and $t \in S_n$, and assume that we have

$$\mathcal{G}_t^+ = \bigcup_{\sigma \in k^n} J^+(t,\sigma) \cap \mathcal{A}_t.$$

We need to show that

$$\mathcal{G}_t^+ = \bigcup_{\tau \in k^{n+1}} I^+(t,\tau) \cap \mathcal{A}_t.$$

But we have

$$\begin{aligned}\bigcup_{\tau \in k^{n+1}} I^+(t,\tau) &= \bigcup_{\sigma \in k^n} \bigcup_{i<k} I^+(t,\sigma^\frown \langle i\rangle) \\ &= \bigcup_{\sigma \in k^n} J^+(t,\sigma) && \text{(by Lemma 101).}\end{aligned}$$

The conclusion now follows easily from the Induction Hypothesis.

[19] We do not really need the full strength of this lemma, though it is one of the elegant results from the ideals being defined the way they are. The problem is that it requires all sequences of a given length, including sequences with repeated colours. The only consequence of this lemma that we shall actually need is the inclusion described in Lemma 105.

Induction step, (2) $\implies$ (1) Fix $n \geq 1$, and assume that for all $s \in S_{n-1}$ we have

$$\mathcal{G}_s^+ = \bigcup_{\sigma \in k^n} I^+(s, \sigma) \cap \mathcal{A}_s.$$

We now fix $t \in S_n$, and we must show that

$$\mathcal{G}_t^+ = \bigcup_{\sigma \in k^n} J^+(t, \sigma) \cap \mathcal{A}_t.$$

Fix $X \in \mathcal{A}_t$. We need to show that

$$X \in \mathcal{G}_t^+ \iff X \in \bigcup_{\sigma \in k^n} J^+(t, \sigma).$$

Since $X \in \mathcal{A}_t$, we can fix some $B \in \mathcal{P}(T) \cap N_t$ such that $X = B \cap t{\downarrow}$. It follows that for every $s \leq_T t$ we have $X \cap s{\downarrow} = B \cap s{\downarrow}$.

Since $t \in S_n$ (where $n \geq 1$), t is a reflection point of S_{n-1}, that is, $S_{n-1} \cap t{\downarrow} \in I_t^+$. By Lemma 96(2), t is also a limit point of S_{n-1}, so that t must also be a limit node of T. By continuity of the nice collection of submodels, we can fix some $r^{\min} <_T t$ such that $B \in N_{r^{\min}}$. So Lemma 95(1) gives $B \cap s{\downarrow} \in \mathcal{A}_s$ for all nodes $s \geq_T r^{\min}$, and in particular, for all nodes $s \in t{\downarrow} \setminus r^{\min}{\downarrow}$.

Using all of these facts, we have

$$\begin{aligned}
& X \in \mathcal{G}_t^+ \\
& \iff X \cap S_{n-1} \in I_t^+ && \text{(Lemma 80)} \\
& \iff X \cap S_{n-1} \setminus r^{\min}{\downarrow} \in I_t^+ && \text{(Lemma 75(4))} \\
& \iff B \cap S_{n-1} \cap \left(t{\downarrow} \setminus r^{\min}{\downarrow}\right) \in I_t^+ && (X = B \cap t{\downarrow}) \\
& \iff \left\{s \in S_{n-1} \cap \left(t{\downarrow} \setminus r^{\min}{\downarrow}\right) : B \cap s{\downarrow} \in \mathcal{G}_s^+\right\} \in I_t^+ && \text{(Lemma 99(2))} \\
& \iff \left\{s \in S_{n-1} \cap \left(t{\downarrow} \setminus r^{\min}{\downarrow}\right) : B \cap s{\downarrow} \in \bigcup_{\sigma \in k^n} I^+(s, \sigma)\right\} \in I_t^+ && \text{(by Ind. Hyp.)} \\
& \iff \bigcup_{\sigma \in k^n} \left\{s \in S_{n-1} \cap \left(t{\downarrow} \setminus r^{\min}{\downarrow}\right) : B \cap s{\downarrow} \in I^+(s, \sigma)\right\} \in I_t^+ \\
& \iff \bigcup_{\sigma \in k^n} \left\{s \in S_{n-1} \cap t{\downarrow} : B \cap s{\downarrow} \in I^+(s, \sigma)\right\} \in I_t^+ && \text{(Lemma 75(4))} \\
& \iff \exists \sigma \in k^n \left(\left\{s \in S_{n-1} \cap t{\downarrow} : B \cap s{\downarrow} \in I^+(s, \sigma)\right\} \in I_t^+\right) \\
& \iff \exists \sigma \in k^n \left(\left\{s \in S_{n-1} \cap t{\downarrow} : X \cap s{\downarrow} \in I^+(s, \sigma)\right\} \in I_t^+\right) \\
& \iff \exists \sigma \in k^n \left(X \in J^+(t, \sigma)\right) \\
& \iff X \in \bigcup_{\sigma \in k^n} J^+(t, \sigma),
\end{aligned}$$

as required. □

Since (for $t \in S_0$) $\mathcal{G}_t$ is a proper ideal in $\mathcal{A}_t \subseteq \mathcal{P}(t{\downarrow})$, Lemma 104 implies the following result: For each $n \geq 0$ and $t \in S_n$, there is some $\sigma \in k^n$ such that $J(t, \sigma)$ is proper; similarly, for each $n \geq 1$ and $t \in S_{n-1}$, there is some $\sigma \in k^n$ such that $I(t, \sigma)$ is proper. When $n = 1$, this gives Corollary 103.

However, for larger n, this fact is not as useful, because there is no way to guarantee that the relevant sequence does not contain repeated colours.

Our main use of Lemma 104 will be the following:

Lemma 105 (cf. [4, Lemma 3.3])**.**

1. *For all $\sigma \in k^{<\omega}$ and all $t \in S_{|\sigma|}$, we have*

$$I_t \subseteq J(t,\sigma),$$

and equivalently,

$$I_t^+ \supseteq J^+(t,\sigma), \text{ and } I_t^* \subseteq J^*(t,\sigma).$$

2. *Similarly, for all nonempty $\sigma \in k^{<\omega}$ and all $t \in S_{|\sigma|-1}$, we have*

$$I_t \subseteq I(t,\sigma),$$

and equivalently,

$$I_t^+ \supseteq I^+(t,\sigma), \text{ and } I_t^* \subseteq I^*(t,\sigma).$$

Proof.

1. Fix $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|}$. From Lemma 104(1), we see that $\mathcal{G}_t \subseteq J(t,\sigma)$. But $\mathcal{G}_t$ is a generating set for the ideal I_t on $t{\downarrow}$. Since $J(t,\sigma)$ is an ideal on $t{\downarrow}$, it follows that $I_t \subseteq J(t,\sigma)$, as required.

2. Fix nonempty $\tau \in k^{<\omega}$ and $t \in S_{|\tau|-1}$. We write $\tau = \sigma^\frown \langle i\rangle$ for some $\sigma \in k^{|\tau|-1}$ and $i < k$. We then have

$$\begin{aligned} I_t &\subseteq J(t,\sigma) && \text{(from part (1))} \\ &\subseteq I(t,\sigma^\frown \langle i\rangle) && \text{(from Lemma 101)} \\ &= I(t,\tau), \end{aligned}$$

as required. □

Lemma 105 will be used several times in what follows.

The following lemma is of slight interest in characterizing the intersections of the ideals with $\mathcal{A}_t$ in the case that they are proper, though we shall not particularly need to use it:

Lemma 106. *For all $\sigma \in k^{<\omega}$ and all $t \in S_{|\sigma|}$, either $J(t,\sigma) = \mathcal{P}(t{\downarrow})$ or $J(t,\sigma) \cap \mathcal{A}_t = \mathcal{G}_t$. Similarly, for all nonempty $\sigma \in k^{<\omega}$ and all $t \in S_{|\sigma|-1}$, either $I(t,\sigma) = \mathcal{P}(t{\downarrow})$ or $I(t,\sigma) \cap \mathcal{A}_t = \mathcal{G}_t$.*

Proof. Let K be either $J(t,\sigma)$ or $I(t,\sigma)$ for some fixed t and σ satisfying the relevant hypotheses. By Lemma 100, K is an ideal on $t{\downarrow}$. By Lemma 70 we know that $\mathcal{A}_t$ is a set algebra over $t{\downarrow}$, so it follows that $K \cap \mathcal{A}_t$ is an ideal in $\mathcal{A}_t$. By Lemma 104 we have $\mathcal{G}_t \subseteq K \cap \mathcal{A}_t$. But Lemma 72 tells us that either $\mathcal{G}_t = \mathcal{A}_t$ or $\mathcal{G}_t$ is a maximal proper ideal in $\mathcal{A}_t$. So it follows that $K \cap \mathcal{A}_t$ must equal either $\mathcal{G}_t$ or $\mathcal{A}_t$. In the first case we are done. In the second case, we have $t{\downarrow} \in \mathcal{A}_t = K \cap \mathcal{A}_t \subseteq K$, so that $K = \mathcal{P}(t{\downarrow})$, and we are done. □

Lemma 107 (cf. [4, Lemma 3.4])**.**

1. *Fix $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|}$. If $X \subseteq t{\downarrow}$ and $X \in J^+(t,\sigma)$, then for all $j \in \operatorname{range}(\sigma)$ there is a j-homogeneous chain $W \in [X]^\kappa$.*

2. *Fix nonempty $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|-1}$. If $X \subseteq t{\downarrow}$ and $X \in I^+(t,\sigma)$, then for all $j \in \operatorname{range}(\sigma)$ there is a j-homogeneous chain $W \in [X]^\kappa$.*

Proof. We prove parts (1) and (2) jointly by induction over the length of the sequence σ.

Base case for (1) If $\sigma = \langle\rangle$ then $\operatorname{range}(\sigma) = \emptyset$ so there is nothing to show.

Induction step, (1) $\implies$ (2) Fix $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|}$, and assume that for all $Z \in J^+(t,\sigma)$ and all $j \in \operatorname{range}(\sigma)$ there is $W \subseteq Z$ such that $|W| = \kappa$ and W is j-homogeneous. We then fix $i < k$, $X \in I^+(t, \sigma^\frown \langle i\rangle)$, and $j \in \operatorname{range}(\sigma^\frown \langle i\rangle)$, and we must find $W \subseteq X$ such that $|W| = \kappa$ and W is j-homogeneous.

Since $X \in I^+(t, \sigma^\frown \langle i\rangle)$, we have $X \cap c_i(t) \in J^+(t,\sigma)$.

There are two cases to consider:

- $j \in \operatorname{range}(\sigma)$: Since $X \cap c_i(t) \in J^+(t,\sigma)$, we use the Induction Hypothesis to find $W \subseteq X \cap c_i(t)$ such that $|W| = \kappa$ and W is j-homogeneous. But then $W \subseteq X$ and we are done.
- $j = i$: From Lemma 105(1) we have $J^+(t,\sigma) \subseteq I_t^+$, so it follows that $X \cap c_i(t) \in I_t^+$. Applying Lemma 86 to the set $X \cap c_i(t)$, we get i-homogeneous $W \subseteq X \cap c_i(t)$ of size κ, as required.

Induction step, (2) $\implies$ (1) Fix nonempty $\sigma \in k^{<\omega}$ and assume that for all $s \in S_{|\sigma|-1}$ and all $Z \subseteq s{\downarrow}$ such that $Z \in I^+(s,\sigma)$ and all $j \in \operatorname{range}(\sigma)$ there is $W \subseteq Z$ such that $|W| = \kappa$ and W is j-homogeneous. We then fix $t \in S_{|\sigma|}$, $X \in J^+(t,\sigma)$, and $j \in \operatorname{range}(\sigma)$, and we must find $W \subseteq X$ such that $|W| = \kappa$ and W is j-homogeneous.

Since $X \in J^+(t,\sigma)$, we have

$$\left\{s \in S_{|\sigma|-1} \cap t{\downarrow} : X \cap s{\downarrow} \in I^+(s,\sigma)\right\} \in I_t^+.$$

In particular, this set, being in the co-ideal I_t^+, must be non-empty. So we fix $s \in S_{|\sigma|-1} \cap t{\downarrow}$ such that $X \cap s{\downarrow} \in I^+(s,\sigma)$. Then we use the Induction Hypothesis to find $W \subseteq X \cap s{\downarrow}$ such that $|W| = \kappa$ and W is j-homogeneous. But then $W \subseteq X$ and we are done. □

Until here, we have focused on describing co-ideals from which we can extract homogeneous sets of order-type κ. Ultimately we shall fix an ordinal $\xi < \log \kappa$, and our strategy will be to find some node $s \in T$ and chains $W, Y \subseteq T$ such that

$$W <_T \{s\} <_T Y,$$

where W has order type κ, Y has order type ξ, and $W \cup Y$ is homogeneous for the colouring c. We shall now work on building structures from which we shall be able to extract homogeneous sets of order-type ξ.

Definition 24. *For any ordinal ρ and sequence $\sigma \in k^{<\omega}$, we consider chains in T of order type $\rho^{|\sigma|}$, and we define, by recursion over the length of σ, what it means for such a chain to be (ρ,σ)-good:*

- *Beginning with the empty sequence $\langle\rangle$, we say that every singleton set is $(\rho, \langle\rangle)$-good.*

- *Fix a sequence $\sigma \in k^{<\omega}$, and suppose we have already decided which chains in T are (ρ, σ)-good. Fix a colour $i < k$. We say that a chain $X \subseteq T$ of order type $\rho^{|\sigma|+1}$ is $(\rho, \sigma^\frown \langle i \rangle)$-good if*

$$X = \bigcup_{\eta < \rho} X_\eta,$$

where the sequence $\langle X_\eta : \eta < \rho \rangle$ satisfies the following conditions:

1. *for each $\eta < \rho$, the chain X_η is (ρ, σ)-good,*
2. *for each $\iota < \eta < \rho$, we have*[20] *$X_\iota <_T X_\eta$, and*
3. *for each $\iota < \eta < \rho$,*

$$c'' (X_\iota \otimes X_\eta) = \{i\}.$$

that is, for each $s \in X_\iota$ and $t \in X_\eta$, we have $c(\{s, t\}) = i$.

Lemma 108 (cf. [4, Lemma 3.5])**.** *Fix $\sigma \in k^{<\omega}$ and ordinal ρ. If a chain $X \subseteq T$ is (ρ, σ)-good, then for all $j \in \mathrm{range}(\sigma)$ there is $Y \subseteq X$ such that Y is j-homogeneous for c and has order-type ρ.*

Proof. By induction over the length of the sequence σ.

Base case If $\sigma = \langle \rangle$ then $\mathrm{range}(\sigma) = \emptyset$ so there is nothing to show.

Induction step Fix $\sigma \in k^{<\omega}$ and assume that for every (ρ, σ)-good set Z and all $j \in \mathrm{range}(\sigma)$ there is $Y \subseteq Z$ such that Y has order type ρ and Y is j-homogeneous. We then fix $i < k$, $(\rho, \sigma^\frown \langle i \rangle)$-good set X, and $j \in \mathrm{range}(\sigma^\frown \langle i \rangle)$, and we must find $Y \subseteq X$ such that Y has order type ρ and Y is j-homogeneous.

There are two cases to consider:

- $j \in \mathrm{range}(\sigma)$: By definition of X being $(\rho, \sigma^\frown \langle i \rangle)$-good, X includes some set X_0 that is (ρ, σ)-good. Then by the Induction Hypothesis, there is $Y \subseteq X_0$ with order type ρ that is j-homogeneous. But then $Y \subseteq X$ and we are done.
- $j = i$: We decompose X into its component subsets X_η, $\eta < \rho$. For each $\eta < \rho$, choose $\gamma_\eta \in X_\eta$. Then the set

$$Y = \langle \gamma_\eta \rangle_{\eta < \rho}$$

is i-homogeneous and satisfies the required conditions. □

Lemma 109 (cf. [4, Lemma 3.6])**.**

1. *Fix $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|}$. If $X \in J^+(t, \sigma)$ then for all $\rho < \kappa$ there is $Y \subseteq X$ that is (ρ, σ)-good.*
2. *Fix nonempty $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|-1}$. If $X \in I^+(t, \sigma)$ then for all $\rho < \kappa$ there is $Y \subseteq X$ that is (ρ, σ)-good.*

Proof. Fix any ordinal $\rho < \kappa$. We prove parts (1) and (2) jointly by induction over the length of the sequence σ.

Base case for (1) If $X \in J^+(\sigma, \langle \rangle)$ then X is certainly nonempty, so choose any $u \in X$, so that $\{u\}$ is $(\rho, \langle \rangle)$-good.

[20] The definition in [4, p. 6] uses $\sup X_\iota < \inf X_\eta$, which is slightly stronger but seems not to be necessary.

Induction step, (1) $\implies$ (2) Fix $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|}$, and assume that for all $Z \in J^+(t, \sigma)$ there is $W \subseteq Z$ such that W is (ρ, σ)-good. We then fix $i < k$, and $X \in I^+(t, \sigma^\frown \langle i \rangle)$, and we must find $Y \subseteq X$ that is $(\rho, \sigma^\frown \langle i \rangle)$-good.

Since $X \in I^+(t, \sigma^\frown \langle i \rangle)$, we have $X \cap c_i(t) \in J^+(t, \sigma)$.

We shall recursively construct a sequence $\langle Y_\eta : \eta < \rho \rangle$ of subsets of $X \cap c_i(t)$ that satisfies the requirements for its union to be $(\rho, \sigma^\frown \langle i \rangle)$-good.

Fix an ordinal $\eta < \rho$ and assume that we have constructed a sequence $\langle Y_\iota : \iota < \eta \rangle$ satisfying the required properties. We show how to construct Y_η.

Let

$$V = \bigcup_{\iota < \eta} Y_\iota .$$

Since $\eta < \rho < \kappa$, $|\sigma|$ is finite, κ is infinite, and for each $\iota < \eta$ we have $|Y_\iota| = \left|\rho^{|\sigma|}\right|$, it follows that

$$|V| = \left| \bigcup_{\iota < \eta} Y_\iota \right| = \sum_{\iota < \eta} |Y_\iota| = \sum_{\iota < \eta} \left|\rho^{|\sigma|}\right| = |\eta| \cdot \left|\rho^{|\sigma|}\right| < \kappa.$$

Of course $V \subseteq t{\downarrow} \subseteq N_t$, so that $V \in [N_t]^{<\kappa}$. Since $t \in S_{|\sigma|} \subseteq S_0$, we have $[N_t]^{<\kappa} \subseteq N_t$, giving us $V \in N_t$.

Define

$$B = \{u \in T : (\forall s \in V) [s <_T u \text{ and } c\{s, u\} = i]\} .$$

Since B is defined from parameters T, V, c, and i that are all in N_t, it follows by elementarity of N_t that $B \in N_t$.

Since $V \subseteq c_i(t)$, it follows from the definition of B that $t \in B$. But then we have $B \cap t{\downarrow} \in \mathcal{G}_t^* \subseteq I_t^*$. By Lemma 105(1), we then have $B \cap t{\downarrow} \in J^*(t, \sigma)$. Recall that $X \cap c_i(t) \in J^+(t, \sigma)$. The intersection of a filter set and a co-ideal set must be in the co-ideal, so we have $B \cap X \cap c_i(t) \in J^+(t, \sigma)$. We now apply the Induction Hypothesis, obtaining (ρ, σ)-good

$$Y_\eta \subseteq B \cap X \cap c_i(t).$$

Since $Y_\eta \subseteq B$, we clearly have $V <_T Y_\eta$ and $c''(V \otimes Y_\eta) = \{i\}$, as required, and we have completed the recursive construction.

We now let

$$Y = \bigcup_{\eta < \rho} Y_\eta$$

so that $Y \subseteq X$ is $(\rho, \sigma^\frown \langle i \rangle)$-good, as required.

Induction step, (2) $\implies$ (1) Fix nonempty $\sigma \in k^{<\omega}$ and assume that for all $s \in S_{|\sigma|-1}$ and all $Z \subseteq s{\downarrow}$ such that $Z \in I^+(s, \sigma)$ there is $Y \subseteq Z$ such that Y is (ρ, σ)-good. We then fix $t \in S_{|\sigma|}$ and $X \in J^+(t, \sigma)$, and we must find $Y \subseteq X$ that is (ρ, σ)-good.

Since $X \in J^+(t, \sigma)$, we have

$$\left\{ s \in S_{|\sigma|-1} \cap t{\downarrow} : X \cap s{\downarrow} \in I^+(s, \sigma) \right\} \in I_t^+.$$

In particular, this set, being in the co-ideal I_t^+, must be non-empty. So we fix $s \in S_{|\sigma|-1} \cap t{\downarrow}$ such that $X \cap s{\downarrow} \in I^+(s, \sigma)$. Then we use the Induction Hypothesis to find (ρ, σ)-good $Y \subseteq X \cap s{\downarrow}$. But then $Y \subseteq X$ and we are done. $\square$

Lemma 110 (cf. [4, Lemma 3.7]). *Fix $\sigma \in k^{<\omega}$ and*[21] *$m < \omega$. If ρ and ξ are any two ordinals such that*

$$\rho \to (\xi)^1_m \,,$$

if $X \subseteq T$ is (ρ, σ)-good, and $g : X \to m$ is some colouring, then there is some $Y \subseteq X$, homogeneous for g, such that Y is (ξ, σ)-good.

Proof. Fix $m < \omega$ and ordinals ρ and ξ satisfying the hypothesis. We prove the lemma by induction over the length of the sequence σ.

Base case If X is $(\rho, \langle\rangle)$-good, then it is a singleton. Any colouring g on a singleton must go to only one colour, so X is homogeneous for g, and being a singleton it is also $(\xi, \langle\rangle)$-good.

Induction step Fix $\sigma \in k^{<\omega}$ and assume that for every (ρ, σ)-good $Z \subseteq T$ and colouring $g : Z \to m$ there is a (ξ, σ)-good $W \subseteq Z$ homogeneous for g. We then fix a colour $i < k$, $(\rho, \sigma^\frown \langle i\rangle)$-good $X \subseteq T$, and a colouring $g : X \to m$, and we must find some $(\xi, \sigma^\frown \langle i\rangle)$-good $Y \subseteq X$ that is homogeneous for g.

Since X is $(\rho, \sigma^\frown \langle i\rangle)$-good, we fix a sequence $\langle X_\eta : \eta < \rho\rangle$ satisfying the conditions in the definition for

$$X = \bigcup_{\eta<\rho} X_\eta$$

to be $(\rho, \sigma^\frown \langle i\rangle)$-good.

Consider any $\eta < \rho$. The set X_η is (ρ, σ)-good, so we apply the Induction Hypothesis to X_η and the restricted colouring $g \restriction X_\eta : X_\eta \to m$. This gives us (ξ, σ)-good $Y_\eta \subseteq X_\eta$ and a colour $j_\eta < m$ such that $g''Y_\eta = \{j_\eta\}$.

Now for each colour $j < m$, define the set

$$V_j = \{\eta < \rho : j_\eta = j\} \,.$$

We now have a partition

$$\rho = \bigcup_{j<m} V_j,$$

so we can fix some $j < m$ and a set $H \subseteq V_j$ of order type ξ.

Now set

$$Y = \bigcup_{\eta\in H} Y_\eta.$$

It is clear that $Y \subseteq X$ is $(\xi, \sigma^\frown \langle i\rangle)$-good and j-homogeneous for g. $\square$

Lemma 111. *Fix $m < \omega$. For any infinite cardinal τ, and any ordinal $\xi < \tau$, there is some ordinal ρ with $\xi \le \rho < \tau$ such that*

$$\rho \to (\xi)^1_m \,.$$

[21] This lemma remains true with m replaced by any cardinal, but we need only the finite case.

Proof. To see this, consider two cases:

- Suppose $\tau = \omega$. In this case, $\xi < \tau$ is necessarily finite, and we have

$$(\xi - 1) \cdot m + 1 \to (\xi)^1_m$$

so we can let $\rho = (\xi - 1) \cdot m + 1$.

- Otherwise, τ is an uncountable cardinal. (This is the case assumed in [4, Lemma 3.7].) For $\xi < \tau$, let $\rho = \omega^\xi$ (where the operation here is ordinal exponentiation). We clearly have $\xi \leq \rho < \tau$. Any ordinal power of ω is *indecomposable*, that is,

$$(\forall m < \omega) \left[\omega^\xi \to \left(\omega^\xi\right)^1_m\right],$$

giving us a homogeneous chain even longer than required. □

From here onward, we shall generally be working within the subtree

$$S_\omega = \bigcap_{n<\omega} S_n,$$

as defined earlier. Notice that if $t \in S_\omega$, then (because $S_\omega \subseteq S_n$ for all $n < \omega$) $I(t, \sigma)$ and $J(t, \sigma)$ are defined for all $\sigma \in k^{<\omega}$ (provided $\sigma \neq \emptyset$ for defining $I(t, \sigma)$).

Also, rather than considering all possible finite sequences of colours $\sigma \in k^{<\omega}$, we shall consider only those sequences that are:

- non-empty (to ensure that we can obtain a homogeneous set of some colour), and
- one-to-one (distinct colours; without repetition — to ensure that its length cannot be longer than the number of colours, so that the collection of such sequences is finite).[22]

Definition 25. *We begin by defining*

$$\Sigma_0 = \left\{\sigma \in k^{<\omega} : \sigma \neq \emptyset \textit{ and } \sigma \textit{ is one-to-one}\right\}.$$

For a stationary subtree $S \subseteq S_\omega$ and $t \in S$, define

$$\Sigma(t, S) = \left\{\sigma \in \Sigma_0 : S \cap t{\downarrow} \in I^+(t, \sigma)\right\}.$$

For any $\sigma \in \Sigma_0$ it is clear that $1 \leq |\sigma| \leq k$. We then have

$$|\Sigma_0| = k + k(k-1) + \cdots + k!$$

which is finite. Since for any t, S we have $\Sigma(t, S) \subseteq \Sigma_0$, there are only finitely many distinct sets $\Sigma(t, S)$.

Lemma 112. *For any stationary $R, S \subseteq S_\omega$, if $t \in R \subseteq S$ then $\Sigma(t, R) \subseteq \Sigma(t, S)$.*

[22]We could have started from the beginning by allowing only sequences without repeated colours in the definition of $I(t, \sigma)$ and $J(t, \sigma)$. Some of the lemmas as stated would be problematic, such as Lemmas 101 and 104, but they are the ones whose full strength we are not using anyway.

Proof. If $R \subseteq S$ then certainly $R \cap t{\downarrow} \subseteq S \cap t{\downarrow}$. For any sequence $\sigma \in \Sigma_0$, we then have

$$\begin{aligned} \sigma \in \Sigma(t, R) &\implies R \cap t{\downarrow} \in I^+(t, \sigma) \\ &\implies S \cap t{\downarrow} \in I^+(t, \sigma) \\ &\implies \sigma \in \Sigma(t, S), \end{aligned}$$

as required. □

For any stationary subtree $S \subseteq S_\omega$, recall that t is called a reflection point of S if $S \cap t{\downarrow} \in I_t^+$. Also recall that by Lemma 97, we have

$$\{t \in S : S \cap t{\downarrow} \in I_t\} \in NS_\nu^T.$$

Lemma 113. *Fix any stationary subtree $S \subseteq S_\omega$. For any $t \in S$, the following are equivalent:*

1. $S \cap t{\downarrow} \in I_t^+$;
2. *There is some colour $i < k$ such that $\langle i \rangle \in \Sigma(t, S)$;*
3. $\Sigma(t, S) \neq \emptyset$.

It follows that

$$\{t \in S : \Sigma(t, S) = \emptyset\}$$

must be a nonstationary subtree

Proof.

(1) $\implies$ (2) Let t be any reflection point of S. We have

$$S \cap t{\downarrow} \in I_t^+ = \bigcup_{i<k} I^+(t, \langle i \rangle)$$

by Corollary 102. So there is some colour $i < k$ such that $S \cap t{\downarrow} \in I^+(t, \langle i \rangle)$. But then $\langle i \rangle \in \Sigma(t, S)$, as required.

(2) $\implies$ (3) Clear.

(3) $\implies$ (1) Suppose $\Sigma(t, S) \neq \emptyset$, and choose $\sigma \in \Sigma(t, S)$. So $S \cap t{\downarrow} \in I^+(t, \sigma)$. Then Lemma 105 gives $S \cap t{\downarrow} \in I_t^+$, as required.

The final statement then follows from Lemma 97. □

For any stationary subtree, Lemma 113 tells us that "almost all" of its points have nonempty Σ, but we should like to have a large set on which Σ is constant. Only the case $R_0 = S_\omega$ of the following lemma will be used in our proof of the Main Theorem 29, but there is no extra effort in stating it with greater generality:

Lemma 114 (cf. [4, Lemma 3.8])**.** *For every stationary subtree $R_0 \subseteq S_\omega$, there are a stationary subtree $R \subseteq R_0$ and a fixed $\Sigma \subseteq \Sigma_0$ such that for all stationary $S \subseteq R$ we have*

$$\{t \in S : \Sigma(t, S) \neq \Sigma\} \in NS_\nu^T.$$

Proof. Fix a stationary subtree $R_0 \subseteq S_\omega$, and recall that Σ_0 is defined previously.

We shall attempt to construct, recursively, decreasing sequences

$$R_0 \supseteq R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots \text{ and } \Sigma_0 \supsetneqq \Sigma_1 \supsetneqq \Sigma_2 \supsetneqq \Sigma_3 \supsetneqq \cdots$$

satisfying the following properties for all $n \geq 0$:

1. R_n is stationary; and

2. for all $t \in R_n$, we have[23] $\Sigma(t, R_n) \subseteq \Sigma_n$

When $n = 0$, we see that R_0 and Σ_0 satisfy the required properties because R_0 was chosen to be stationary and every possible $\Sigma(t, R_0)$ is a subset of Σ_0.

Fix $n \geq 0$, and assume we have constructed R_n and Σ_n satisfying the requirements. We attempt to choose R_{n+1} and Σ_{n+1}, as follows:

Consider any stationary set $S \subseteq R_n$. For each $\Gamma \subseteq \Sigma_n$ define

$$S^\Gamma = \{t \in S : \Sigma(t, S) = \Gamma\}.$$

There are now two possibilities:

- If there is some stationary $S \subseteq R_n$ and $\Gamma \subsetneqq \Sigma_n$ such that S^Γ is stationary, then let $\Sigma_{n+1} = \Gamma$ and $R_{n+1} = S^\Gamma$. For each $t \in R_{n+1}$, since $R_{n+1} \subseteq S$, we have (using Lemma 112)

  $$\Sigma(t, R_{n+1}) \subseteq \Sigma(t, S) = \Gamma = \Sigma_{n+1},$$

  so it is clear that R_{n+1} and Σ_{n+1} satisfy the properties required for our decreasing sequences.

  Recall that Σ_0 is finite. A strictly decreasing sequence of subsets of a finite set cannot be infinite, so after some finite m, this alternative will be impossible.

- Otherwise, for all stationary $S \subseteq R_n$ and all $\Gamma \subsetneqq \Sigma_n$, S^Γ is nonstationary. So we set $R = R_n$ and $\Sigma = \Sigma_n$ and we verify that these sets satisfy the conclusion of the lemma:

  Fix a stationary subtree $S \subseteq R_n$. For any $t \in S$, Lemma 112 and property (2) above give

  $$\Sigma(t, S) \subseteq \Sigma(t, R_n) \subseteq \Sigma_n,$$

  so that we have

  $$\{t \in S : \Sigma(t, S) \neq \Sigma_n\} = \bigcup_{\Gamma \subsetneqq \Sigma_n} S^\Gamma.$$

  There are only finitely many subsets of Σ_n, so this set is is the union of finitely many nonstationary subtrees, so it must be nonstationary, as required. □

From Lemma 113, it follows that any Σ obtained from Lemma 114 must be nonempty. Since any $\Sigma \subseteq \Sigma_0$ is also finite, it is reasonable to consider a sequence of colours $\sigma \in \Sigma$ that is maximal by inclusion. Here we explore the consequences of σ being maximal.

[23] This condition was misstated in [4] and [2], leading to some confusion.

Lemma 115 (cf. [4, Lemma 3.9]). *Suppose $S \subseteq S_\omega$ is stationary, and there is $\Sigma \subseteq \Sigma_0$ such that*

$$\{t \in S : \Sigma(t, S) \neq \Sigma\} \in NS_\nu^T.$$

Suppose also that $\sigma \in \Sigma$ is maximal by inclusion. Then there are $s \in S$ with $\Sigma(s, S) = \Sigma$ and stationary $R \subseteq S$, with $\{s\} <_T R$, such that

$$(\forall t \in R)\left[S \cap s{\downarrow} \cap \bigcup_{i \notin \mathrm{range}(\sigma)} c_i(t) \in I(s, \sigma)\right].$$

Proof. We define

$$S' = \{t \in S : \Sigma(t, S) = \Sigma\}$$

and

$$S'' = \left\{t \in S' : S' \cap t{\downarrow} \in I_t^+\right\}.$$

By hypothesis, S is stationary, and $\{t \in S : \Sigma(t, S) \neq \Sigma\}$ is nonstationary, so S' is stationary. Applying Lemma 97 to S' gives us that $\{t \in S' : S' \cap t{\downarrow} \in I_t\}$ is nonstationary, so it follows that S'' is stationary.

By assumption, σ is maximal in Σ. That is, $\sigma \in \Sigma$ but

$$(\forall i \notin \mathrm{range}(\sigma))\left[\sigma^\frown \langle i \rangle \notin \Sigma\right].$$

Now consider any $t \in S''$. Since $t \in S'$, we have $\Sigma(t, S) = \Sigma$. For every $i \notin \mathrm{range}(\sigma)$, we have $\sigma^\frown \langle i \rangle \notin \Sigma(t, S)$, meaning that $S \cap t{\downarrow} \notin I^+(t, \sigma^\frown \langle i \rangle)$, equivalently $S \cap t{\downarrow} \in I(t, \sigma^\frown \langle i \rangle)$, and $S \cap t{\downarrow} \cap c_i(t) \in J(t, \sigma)$. It follows that

$$\bigcup_{i \notin \mathrm{range}(\sigma)} S \cap c_i(t) \in J(t, \sigma),$$

meaning that

$$\left\{s \in S_{|\sigma|-1} \cap t{\downarrow} : \bigcup_{i \notin \mathrm{range}(\sigma)} S \cap c_i(t) \cap s{\downarrow} \in I^+(s, \sigma)\right\} \in I_t.$$

Since $t \in S''$, we have $S' \cap t{\downarrow} \in I_t^+$. Then, since $S' \subseteq S \subseteq S_\omega \subseteq S_{|\sigma|-1}$, we can choose $s_t \in S' \cap t{\downarrow}$ such that

$$\bigcup_{i \notin \mathrm{range}(\sigma)} S \cap c_i(t) \cap s_t{\downarrow} \in I(s_t, \sigma).$$

So for every $t \in S''$ (a stationary subtree of T), we have chosen $s_t <_T t$ with $s_t \in S'$, satisfying the formula immediately above. This defines a regressive function on a stationary subtree, so by Theorem 47 it must have a constant value $s \in S'$ on some stationary subtree $R \subseteq S''$ with $\{s\} <_T R$, meaning that for all $t \in R$, we have $s_t = s$, giving

$$\bigcup_{i \notin \mathrm{range}(\sigma)} S \cap c_i(t) \cap s{\downarrow} \in I(s, \sigma).$$

Since $R \subseteq S'' \subseteq S' \subseteq S$, this implies the required result. □

Now it's time to put everything together to get the required homogeneous sets. Fix an ordinal[24]

[24] Recall from chapter 2 that $\log \kappa$ is the smallest cardinal τ such that $2^\tau \geq \kappa$.

$\xi < \log \kappa$. Recall that T is a non-ν-special tree (where $\nu = 2^{<\kappa}$), and $c : [T]^2 \to k$, and we need to find a chain $X \subseteq T$ of order type $\kappa + \xi$ that is homogeneous for the partition c.

Recall that S_ω is stationary (Lemma 98(4)). Using Lemma 114, we fix stationary $S \subseteq S_\omega$ and $\Sigma \subseteq \Sigma_0$ such that for all stationary $R \subseteq S$ we have

$$\{u \in R : \Sigma(u, R) \neq \Sigma\} \in NS_\nu^T.$$

Using Lemma 113, $\Sigma \neq \emptyset$. Fix $\sigma \in \Sigma$ that is maximal by inclusion, and let $m = |\sigma|$.

We now apply Lemma 115 to S, Σ, and σ. This gives us $s \in S$ with $\Sigma(s, S) = \Sigma$ and stationary $R \subseteq S$, with $\{s\} <_T R$, such that

$$(\forall u \in R)\left[S \cap s{\downarrow} \cap \bigcup_{i \notin \mathrm{range}(\sigma)} c_i(u) \in I(s, \sigma)\right].$$

Our goal will be to find chains $W \subseteq S \cap s{\downarrow}$ and $Y \subseteq R$ such that W has order-type κ, Y has order-type ξ, and $W \cup Y$ is homogeneous for c. That is, we require the chains W and Y to satisfy

$$[W]^2 \cup (W \otimes Y) \cup [Y]^2 \subseteq c^{-1}(\{i\})$$

for some $i < k$.

Since $\Sigma(s, S) = \Sigma$, we have $\sigma \in \Sigma(s, S)$, meaning

$$S \cap s{\downarrow} \in I^+(s, \sigma).$$

Since $R \subseteq S$, by choice of S we have

$$\{u \in R : \Sigma(u, R) \neq \Sigma\} \in NS_\nu^T,$$

and R is stationary, so we can fix $u \in R$ such that $\Sigma(u, R) = \Sigma$, so that $\sigma \in \Sigma = \Sigma(u, R)$, giving

$$R \cap u{\downarrow} \in I^+(u, \sigma).$$

We have $\xi < \log \kappa \leq \kappa$, where of course $\log \kappa$ is infinite.

We apply Lemma 111 to the ordinal ξ, obtaining an ordinal ρ with $\xi \leq \rho < \log \kappa$ such that

$$\rho \to (\xi)^1_m.$$

We then apply Lemma 109 to $R \cap u{\downarrow}$ and the ordinal ρ, to obtain $Z \subseteq R \cap u{\downarrow}$ that is (ρ, σ)-good. Since $Z \subseteq R$, we have $\{s\} <_T Z$ and for every $t \in Z$ we have

$$S \cap s{\downarrow} \cap \bigcup_{i \notin \mathrm{range}(\sigma)} c_i(t) \in I(s, \sigma).$$

Since Z is (ρ, σ)-good, it has order type ρ^m, and therefore $|Z| = |\rho^m| < \log \kappa \leq \kappa$. Since $I(s, \sigma)$ is a

κ-complete ideal (Lemma 100), it follows that

$$\bigcup_{t \in Z} \left(S \cap s{\downarrow} \cap \bigcup_{i \notin \mathrm{range}(\sigma)} c_i(t) \right) \in I(s, \sigma),$$

or

$$S \cap s{\downarrow} \cap \bigcup_{t \in Z} \left(\bigcup_{i \notin \mathrm{range}(\sigma)} c_i(t) \right) \in I(s, \sigma).$$

We now let

$$H = S \cap s{\downarrow} \setminus \bigcup_{t \in Z} \left(\bigcup_{i \notin \mathrm{range}(\sigma)} c_i(t) \right),$$

and since $S \cap s{\downarrow} \in I^+(s, \sigma)$, it follows that

$$H \in I^+(s, \sigma).$$

We can also write

$$H = \{ r \in S \cap s{\downarrow} : (\forall t \in Z) \, [c(\{r, t\}) \in \mathrm{range}(\sigma)] \}.$$

For each $r \in H$, we define a function $g_r : Z \to \mathrm{range}(\sigma)$ by setting, for each $t \in Z$,

$$g_r(t) = c(\{r, t\}).$$

How many different functions from Z to $\mathrm{range}(\sigma)$ can there be? At most $|\sigma|^{|Z|}$. But $|Z| < \log \kappa$ and σ is finite, so $|\sigma|^{|Z|} < \kappa$.

For each function $g : Z \to \mathrm{range}(\sigma)$, define

$$H_g = \{ r \in H : g_r = g \}.$$

There are fewer than κ such sets, and their union is all of H, which is in the κ-complete co-ideal $I^+(s, \sigma)$, so there must be some function g such that $H_g \in I^+(s, \sigma)$. Fix such a function $g : Z \to \mathrm{range}(\sigma)$.

We then apply Lemma 110 to the colouring g, and we obtain $Z' \subseteq Z$, homogeneous for g, that is (ξ, σ)-good. That is, we have a (ξ, σ)-good $Z' \subseteq Z$ and a fixed colour $i \in \mathrm{range}(\sigma)$ such that for all $t \in Z'$ we have $g(t) = i$. But this means that for all $r \in H_g$ and all $t \in Z'$ we have

$$c(\{r, t\}) = g_r(t) = g(t) = i,$$

showing that $H_g \otimes Z' \subseteq c^{-1}(\{i\})$.

Now Z' is (ξ, σ)-good and $i \in \mathrm{range}(\sigma)$, so using Lemma 108 we fix $Y \subseteq Z'$ that is i-homogeneous for c and has order type ξ.

Also, applying Lemma 107 to H_g, we get $W \subseteq H_g$ such that $|W| = \kappa$ and W is i-homogeneous for c.

So then $W \cup Y$ is i-homogeneous of order type $\kappa + \xi$, as required. This completes the proof of the Main Theorem, Theorem 29.

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