The Left Regular Representation of a Semigroup

by

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Abstract

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As with groups, one can study the left regular representation of a semigroup. If one considers such representations, then it is natural to ask similar questions to the group case.

We start by formulating several questions in the semigroup case and then work towards understanding the structure of the representations given. We present results describing what the elements of the image under the representation map can look like (the semigroup problem), whether or not two semigroups will give isomorphic representations (the isomorphism problem), and whether or not the representation of a semigroup is reflexive (the reflexivity problem).

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Contents

Chapter 1

Introduction

The motivation for this thesis is to understand the structure of the object $L(S)$, the left regular representation of a semigroup S. The case where S is actually a group has been well studied, regarding $L(S)$ as an algebra of operators, and also as a *-algebra.

The semigroup case, however, has not been studied as deeply, possibly due to characters being a less valuable tool for study, and also since $L(S)$ is not closed under taking adjoints. Davidson and Pitts in [1] studied the case of $L(S)$ when S is a countably generated free semigroup (the resulting algebra they regarded as being a non-commutative analytic Toeplitz algebra), and were able to prove its hyper reflexivity, as well as other properties. Hadwin and Nordgren in [3], as particular examples of multiplier algebras, showed that for two large classes of semigroups, the commutant of $L(S)$ is $R(S)$, the right regular representation.

While studying $L(S)$, it is often useful to explicitly calculate what the algebra is. This inspires the question of what exactly the generating operators actually look like. In the group case, we know that they are all unitary operators, moreover, relative to the group element basis, they are direct sums of a number of copies of a finite cycle, or of bilateral shifts of some multiplicity. The problem of classifying these multiplication operators in the semigroup case we call the Semigroup Problem, which we will spend most of Chapter

3 investigating, followed by some additional applications.

While at first glance there is no 'nice' class of operators that these multiplications fit into (they are generally not unitary), there are other properties that these operators satisfy. One such property is a (non-unitary) version of the Wold decomposition which, as we will see in Chapter 4, comes from their relationship to isometries.

Properties of the entire algebra $L(S)$ will also be considered. In Chapter 5 we spend some time considering when $L(S)$ is reflexive. While $L(S)$ will rarely be reflexive itself, the algebra generated by all possible semigroup multiplication operators for a fixed Hilbert space (which we denote UMA) will be. This result is explored more deeply in Chapter 6.

While no definitive results are obtained, Chapter 7 discusses some partial results about the Isometric Problem (a variant of the isomorphism problem we pose for semigroup representations) and include some additional remarks.

Finally, we end the discussion with some open problems.

Chapter 2

Preliminaries

We begin with a semigroup S and define our Hilbert space H by taking square summable sequences indexed by the elements of S, with the usual inner product. When needed, we shall distinguish an element $s \in S$ from its image in $l^2(S)$ by denoting it as e_s. However, when the context is clear we may refer to e_s as just s for convenience.

Since S is a semigroup, we can consider the operators induced by multiplication of each element $s \in S$ by another element $a \in S$.

Definition 2.0.1. *For an element $a \in S$, we define L_a as the operator that sends the basis element $s \in S$ to as. Likewise, we define R_a as the operator which sends s to sa. The algebra $L(S)$ will denote the weakly closed algebra generated by all such operators L_a, and similarly $R(S)$ will denote the weakly closed algebra generated by all the operators R_a.*

A natural question to ask is when L_a (or R_a) is bounded. Hadwin and Nordgren in [3] use the following criterion on the multiplications to deduce boundedness for a particular class of semigroups which we will mention later. Here, we give a proof, as well as estimates of the norm of the multiplication based on the algebraic bound.

Proposition 2.0.1. *The operator L_a is bounded if and only if there exists $n \in \mathbf{N}$ such that for all $t \in S$, $\#\{s \in S \mid as = t\} \leq n$. Moreover, if L_a is bounded, then $\sqrt{n} \leq ||L_a|| \leq n$.*

Proof. Suppose that L_a is bounded, but the first conclusion is false. Then either there is some $t \in S$ for which $\# \{s \in S \mid as = t\}$ is infinite, or there is a sequence $\{t_j\}_{j \in \mathbf{N}}$, for which $\# \{s \in S \mid as = t_j\}$ tends to infinity. Since each $as = t_j$ exactly means that L_a has a one in the s column and t_j row, the above either means that the sum of a row is infinite, or there is a sequence of rows whose sums are increasing infinitely. Since each row has either ones or zeros, this means the square sum is infinite or tending to infinity, and therefore, L_a would have to be unbounded, a contradiction.

For the opposite direction, we assume that there is an n such that for all $t \in S$, $\# \{s \in S \mid as = t\} \leq n$. By the same logic as the above, we have that the square sum of each row is bounded by n. Likewise, we also have that the columns are bounded in their square sum, since they all contain exactly one 1. Thus, L_a must be bounded.

Finally we remark that for a given n, if $\# \{s \in S \mid as = t\} = n$ for only a single t, while all other values of t give a cardinality of 1, then the norm of L_a is $\sqrt{n}$ since one can take the unit vector that is $\frac{1}{\sqrt{n}}$ in the n coordinates that are sent to t under L_a, and the norm of this new vector will be $\sqrt{n}$. Likewise, one could take L_a such that $\# \{s \in S \mid as = t\} = n$ for all t. In this case, for a vector m onto which L_a is applied, at worst the coordinates of m will be multiplied by n. This gives $|| \sum_{j=1}^{\infty} L_a(s_j)|| \leq \sqrt{\sum_{j=1}^{\infty}(ns_j)^2} = \sqrt{n^2 \sum_{j=1}^{\infty}(s_j)^2} = n||s_j||$, making the norm at most n.

□

From the above, it is natural to require the semigroup S we study to have only bounded multiplications. This gives us the definition of a bounded semigroup.

Definition 2.0.2. *A semigroup S will be called* **bounded** *if for any $a \in S$, there exists $n \in \mathbf{N}$ such that for all $t \in S$, $\# \{s \in S \mid as = t\} \leq n$, and $\# \{s \in S \mid sa = t\} \leq n$. That is, for any $a \in S$, L_a and R_a are bounded operators.*

We remark that for a semigroup, the left multiplications being bounded do not imply that the right multiplications are bounded. For example one could take the infinite semigroup S with each left multiplication defined as being the identity. For any a, b in this semigroup, $ab = b$, so each left multiplication is bounded (with a norm of 1), while each right multiplication is unbounded since it will send every basis element to itself.

For most cases, we will not be interested in what the actual norm of the operator L_a is, but rather we care more about what the constant n given above is for L_a. To distinguish between this and the usual norm, we shall refer to this constant as the cardinality norm of L_a.

Definition 2.0.3. *For a bounded left multiplication L_s in a semigroup S, we will call the smallest constant n for which $\#\{s \in S \mid as = t\} \leq n$ for all $t \in S$ as the* **cardinality norm** *of L_a.*

We note that the cardinality norm of L_a is close enough to the actual norm of L_a, since the norm of L_a is between $\sqrt{n}$ and n. However, the individual requirements for the boundedness of the operators L_a doesn't say anything about global properties of the semigroup itself. In some cases, it would be useful to know that the operators we are dealing with are uniformally bounded in some way.

Definition 2.0.4. *For a bounded semigroup S, we call S* **uniformally bounded** *if there is some $n \in \mathbf{N}$ such that all left multiplication operators and all right multiplication operators of S are bounded in cardinality norm by n.*

Hadwin and Nordgren showed that the regular representations of these types of semigroup form generalized multiplier algebras, and in particular, have the property that the commutant of the their left regular representation is the right regular representation [3]. From the earlier remarks about the relationship between the cardinality and the operator norms of L_a, we note that the above definition of uniformally bounded is equivalent to requiring that the operator norms of the L_a's be uniformally bounded.

Some examples of uniformally bounded semigroups include groups and cancellative semigroups (both with a uniform cardinality norm of 1), as well as all finite semigroups.

However, some of these examples also act like another important class of bounded semigroups, which also form generalized multiplier algebras [3].

Definition 2.0.5. *For a bounded semigroup S, we call S* **finite bounded** *if for each $a \in S$, there are only finitely many b's and c's in S such that $a = bc$.*

This condition on the decomposition into factors of semigroup elements in S is similar in nature to the condition on Étale groupoids, except we are dealing with discrete semigroups. Examples of finite bounded semigroups include free semigroups, finite semigroups, and the natural numbers with multiplication as $ab = \max\{a, b\}$. In particular, the latter example shows that a semigroup S that is finite bounded need not be uniformly bounded since L_n will send all natural numbers less than n to n itself, though the first example shows that it can indeed be uniformly bounded since free semigroups are cancellative. As well, we note that an infinite group G is never finite bounded since every element of G appears infinitely many times in the multiplication table of G. Thus, the finite bounded condition on a semigroup should not be confused with any sort of condition on the norms of the multiplication operators of a semigroup.

We note one simple property of finite bounded semigroups.

Proposition 2.0.2. *If S is a finite bounded semigroup with an infinite number of elements, then the number of possible decompositions of the elements of S must not be uniformly bounded.*

Proof. Suppose otherwise that there is some uniform bound n on the number of decompositions of each element. Then we can take distinct $a_0, a_1, ..., a_{n^2+2} \in S$, and consider the product $a = a_0 a_1 ... a_{n^2+2}$. By associativity we have that a can be decomposed into $n^2 + 1$ products. This does not give an immediate contradiction since some of these decompositions could be the same, for example, if we have $a = bcdef$, then $(bcde)f = bcd(ef)$ but ef could equal f and $bcde$ could equal bcd, which would not yield two distinct decompositions.

However, we have $n^2 + 1$ such decompositions, and we can only have at most n distinct ones, so by the pigeon hole principle there must be some multiplication that has more than n decompositions. The factors for this decomposition must be equal. Looking at the terms on the right, this gives us at least $n + 1$ terms that must be equal, each of which having a distinct left most element. This however gives us an element that has at least $n + 1$ distinct decompositions, and so we have a contradiction. Taking an example, suppose n is 2 and we take $2^2 + 2 = 6$ distinct elements in S and call them a, b, c, d, e, f, then the product $abcdef = (abcde)(f) = (abcd)(ef) = (abc)(def) = (ab)(cdef) = (a)(bcdef)$. At least three of these decompositions must be same, so say, $abcd = abc = ab$ and $ef = def = cdef$. Looking at the latter terms, we have that $(e)(f) = (d)(ef) = (c)(def)$, and by the assumption, e,d and c are all distinct, the element ef has at least three distinct decompositions, contradicting the fact that there should be at most $n = 2$. □

2.1 Constructions and Examples

When dealing with bounded semigroups, it will be useful to know some constructions for putting them together to get bigger semigroups which are still bounded. Perhaps the simplest construction that will come to mind is the direct sum, which does preserve boundedness.

Theorem 2.1.1. *Let S and T be semigroups. $S \oplus T$ is defined in the obvious way as ordered pairs from S and T, with multiplication occurring coordinate wise. Boundedness carries over as follows:*

1. *$S \oplus T$ is bounded if and only if S and T are bounded*

2. *$S \oplus T$ is uniformly bounded if and only if S and T are uniformly bounded*

3. *$S \oplus T$ is finite bounded if and only if S and T are finite bounded.*

Proof. Restricting to the subsemigroups corresponding to S and T inside $S \oplus T$, we have the forward direction for all three statements, so all we have to proof is the reverse directions.

1.) $\Leftarrow$), Let (a, b) be in $S \oplus T$, with the cardinality norm of L_a and L_b as n and m respectively. We wish to show that for some natural number p, for all $(s, t) \in S \oplus T$, $\#\{(x, y) \in S \oplus T \mid (ax, by) = (s, t)\} < p$. In order for $(ax, by) = (s, t)$, we must have that $ax = s$ and $by = t$. Since the cardinality norm of L_a is n, we know that there are at most n such x's that will give $ax = s$, and by the cardinality norm of L_b, at most m such y's that will give $by = t$, so there are at most nm such (x, y) to give $(ax, by) = (s, t)$. Thus, since nm is independent of (s, t), we have that the cardinality norm of $L_{(a,b)}$ is at most the cardinality norm of L_a times the cardinality norm of L_b (moreover, by picking s and t appropriately, we can obtain this cardinality norm exactly). Thus $L_{(a,b)}$ is bounded, so $S \oplus T$ is bounded.

2.) $\Leftarrow$), Using the fact that the cardinality norm of $L_{(a,b)}$ is just the multiplication of the cardinality norms on the coordinates, we have that if S and T are uniformly bounded by cardinality norms s and t, then $S \oplus T$ will be uniformly bounded by cardinality norm st.

3.) $\Leftarrow$), We wish to show that for any $(a, b) \in S \oplus T$, the set $\{(s, t, x, y) \in (S \oplus T) \oplus (S \oplus T) \mid (sx, ty) = (a, b)\}$ is finite. Since S and T are finite bounded, we have that there are finitely many (s, x)'s that will give $sx = a$ and likewise finitely many (t, y)'s that will give $ty = b$. The number of ways of getting $(sx, ty) = (a, b)$ is then a finite number times another finite number, which is finite. Thus $S \oplus T$ is finite bounded. □

The if and only if of the above statements now gives us a way of easily constructing some interesting examples of bounded semigroups.

Example 2.1.0.1

Let S be any uniformly bounded semigroup that is not finite bounded, and let T be a finite bounded semigroup that is not uniformly bounded. By the above theorem, $S \oplus T$ will be bounded, but will not be uniformly bounded, nor finitely bounded.

As a concrete example of such a semigroup, one can take S as any infinite group (for example, the integers), and T as the natural numbers with the multiplication defined as $ab = \max\{a, b\}$.

Chapter 3

The Semigroup Problem

Before we start working on showing some stronger properties of the algebra $L(S)$, we first study the properties of some of the individual matrices inside $L(S)$. In particular, we try to understand, in some sense, what kind of structure the matrix of L_a can have for $a \in S$. We know already that the matrix of any L_a will have exactly one 1 in each column and 0's elsewhere. Such a matrix we will call a *basis map matrix* (since it can be thought of as the induced matrix from a map of the basis to itself), while a matrix that arises as a left multiplication in a semigroup will be called a *semigroup matrix.* Stated formally for operators we have,

Definition 3.0.1. *For a given Hilbert space with orthogonal basis, a bounded operator A will be called a* **basis map operator** *if the image of every basis element is another basis element. Moreover, if a semigroup operation can be defined on the basis so that A can be obtained as L_a for some basis element a, then A will be called a* **semigroup operator***.*

We note that the assignment of the semigroup operation on such a basis is equivalent to saying that there is a semigroup whose left regular representation contains the operator A. The semigroup problem now becomes:

Problem 3.0.1. *(The Semigroup Problem) Given a basis map operator A, is there a semigroup operation on the basis for which A is the left multiplication of some basis* *element? Can the semigroup be chosen to be commutative? Unital? Bounded?*

Basis map operators are closed under multiplication, but not all operators of this form can arise from a semigroup (hence the notion of a basis map operator and a semigroup operator are distinct). The simplest example of this is the permutation matrix,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since $A, A^2, ..., A^6$ are all distinct, if this matrix did come from some semigroup as multiplication by a, then $a, a^2, ..., a^6$ would have to be distinct, which contradicts the fact that the cardinality of S must be the same as the dimension of H.

There isn't an easy fix for this problem by adding additional semigroup elements so that the order of A is less than or equal to the the dimension of the Hilbert space. For example, one might consider taking the above A and embedding it into the 6×6 matrices by just expanding A by a row and column and making it act like the identity on the new basis vector, but this will not work.

To see this in the example above, we give a different proof that will illustrate the problem more clearly. Suppose that the semigroup elements are $a_0, ..., a_4$, with the first column of A corresponding to a_0, the second to a_1 and so on. Now, the elements a_0 and a_1 form the two cycle of A, the elements a_2, a_3, a_4 form the three cycle of A, and since we've assumed that A comes from a semigroup, we also have that A is left multiplication by a_m for one of the basis elements a_m. The problem that occurs is that either $A^3 = A$ (if a_m is part of the two cycle) or $A^4 = A$ (if a_m is part of the three cycle). If A is left multiplication by a_0, then $a_0^3 = a_0(a_0^2) = a_0(a_1) = a_0$ thus $L_{a_0}^3 = L_{a_0}$, but A is of order 6, so this is impossible. Likewise, $A \neq L_{a_1}$ and similarly for a_2, a_3, and a_4.

This proof depends only on the order of A and the orders of its cycles, and not on the size of the semigroup, so embedding A into higher dimensions (in an easy way[1]) will never make it a matrix arising from a semigroup.

3.1 Cancellative Basis Map Operators

To start on the semigroup problem, we will first restrict our attention to basis map operators that are injective, or, in terms of their semigroup multiplication, left cancellative. The fact that a basis map operator is injective as an operator on a Hilbert space is equivalent to it being injective on just the basis is easy to establish.

Proposition 3.1.1. *For a basis map operator A, A is injective if and only if it is injective when restricted to its basis.*

Proof. The forward direction is obvious, so we show the reverse. We will show that $\ker(A) = 0$. For any vector x, we know that

$$A(x) = \begin{pmatrix} A_1 \cdot x \\ A_2 \cdot x \\ A_3 \cdot x \\ \vdots \end{pmatrix}$$

where A_i is the i-th row of A. Since A is a basis map operator, we know that every column of A has exactly one 1, and since it is injective on the basis, we know that it has exactly one 1 in each row. Using this, we wish to show that for each i, the i-th coordinate of x is zero. To do this, we consider $A_{A(e_i)} \cdot x = 0$ (this makes sense since A is a basis map operator). Since $A_{A(e_i)}$ is the $A(e_i)$-th row of A, it has zeros everywhere except at i-th position, which is 1, and so $A_{A(e_i)}x = 0$ implies that $1 \cdot x_i + 0 \cdot x_1 + 0 \cdot x_2 + ... = 0$,

[1]For the given example there will be a way of embedding it into a higher dimensional space so that it's restriction to a subspace is A and A is a left multiplication matrix. This fact will explored in at the end of this chapter.

and so $x_i = 0$. Since i was an arbitrary coordinate, we have that x must be 0, and so A is injective as an operator. □

In the future, we will ignore the distinction between injectivity on the basis and injectivity of the operator and just say that the basis map operator is injective.

Injective basis map operators will be used as building blocks for more general basis map operators later, and the constructions used in this case will be extended later to include the general cases. We first note though that solving the Semigroup Problem for cancellative basis map operators is different from classifying multiplications on cancellative semigroups, since we only require that a single left multiplication be cancellative, and not the whole semigroup. Indeed, in the finite semigroup case, any cancellative semigroup will actually be a group, while there are many examples of semigroups that have only a few multiplications being cancellative that are not groups.

Here we have two main cases to deal with: when A is a finite dimensional operator (giving us a permutation) and when A is infinite dimensional.

3.1.1 Cycle Decomposition

The most important structure of cancellative basis map operators that we will use is its decomposition into what we call cycles. For a finite dimensional permutation matrix, this gives us a direct sum of cycles of finite length. In the infinite case, we get additional cases.

Definition 3.1.1. *For a cancellative basis map operator A, we will call a subset M of the basis a* **cycle** *(for A) if M has an element y such that M is the union of all the images and preimages of y under powers of A.*

If M is a finite set with n elements, we call M an n-**cycle**. *If $A|_M$ is the unilateral shift, we call M an* **N cycle**, *and if $A|_M$ is the bilateral shift we call it a* **Z cycle**.

Lastly, for a cycle M of A, we call $A|_M$ a cycle part of A.

The definition of a cycle for non-cancellative basis map operators will be slightly different than in the above, but this will be mentioned later in section 3.2.1.

Here we note that a **Z** cycle will contain many cyclic subsets on which A looks like the unilateral shift instead of the bilateral one. For these cases it will be better for us to just consider only maximal subsets instead, and hence the requirement that M be closed under taking *preimages* under A as well.

Our next goal will be to decompose cancellative basis map operators according to their cycles. First however, we will need some results regarding the structure of these operators.

Lemma 3.1.1. *Let A be a cancellative basis map operator. If a cycle of A is finite, then the cycle part of A on this cycle is equivalent (under a permutation of the basis) to the matrix with ones just below the main diagonal, a one in the top right corner, and zeros elsewhere. If a cycle of A is infinite, then the cycle part of A for this cycle is either a unilateral shift, or a bilateral shift. Thus, the types of cycles listed in the above definition are the only possibilities.*

Proof. If a cycle is finite for A, then as we take images of an element in the cycle, we must eventually come back to the element we started with (since A is injective). Reordering the basis with the first element, then its image under A, A^2, and so on, gives us a matrix with the desired description.

If a cycle of A is infinite, then we have two possible cases. Either for the given element y in M we can keep taking preimages under A, or after a while we can't take any more. If we can always keep taking preimages under A, then we can identify the element y as 0, $A(y)$ as 1, and so on, while the preimage of y as -1, its preimage as -2 and so on. On these elements, multiplication by A acts as addition by 1 on these elements, and so $A|_M$ is the bilateral shift.

If after a while we can can't take any more preimages, then we identify the element that is furthest back as 0, it's image as 1, and so on. On these elements A acts as like addition by 1, and so $A|_M$ is the unilateral shift. □

Lemma 3.1.2. *Let A be a cancellative basis map operator. Then the span of any cycle of A is a reducing subspace of A.*

Proof. Since any cycle is closed under taking images of A, its span is an invariant subspace of A. As well, since no basis element in the complement of the cycle goes into the cycle (since A is cancellative and a cycle includes all possible preimages), all the basis elements in the complement of the cycle are sent into the complement, and so the orthogonal complement of the subspace is also invariant, and hence any cycle of A is a reducing subspace of A. □

Lemma 3.1.3. *Every basis element of a cancellative basis map operator A is in some cycle of A. Moreover, if two cycles M and N of A have a non-empty intersection, then $M = N$.*

Proof. Let y be any basis element. Let the set M consist of all images of y under powers of A and all preimages of y under A. This M forms a cycle with $y \in M$, so the first part is shown.

Next, suppose that there are two cycles M and N, with elements x and y that generate the M and N cycles respectively. If $z \in M \cap N$, then from x, we can either take images or preimages and eventually get to z, and likewise for y. This means that from z we can take either take images or preimages to get to both x and y. Thus any image or preimage of x is either an image or preimage of z and likewise for y, and thus both M and N must contain the same elements, and so M and N must actually be the same cycle. □

With the above, we can now decompose infinite cancellative basis map operators into a direct sum of their cycle parts.

Theorem 3.1.1. *(Cancellative Basis Map Operator Cycle Decomposition) For any cancellative basis map operator A on a Hilbert space H, A can be written as a direct sum of finite, $\mathbf{N}$, and $\mathbf{Z}$ cycle parts.*

Proof. Let $\mathcal{F}$ be the family of all collections of disjoint cycles of A, with a partial ordering defined by inclusion. If $\{\mathcal{D}_\alpha\}$ is a totally ordered subfamily of $\mathcal{F}$, and $\mathcal{D} = \cup_\alpha\{\mathcal{D}_\alpha\}$, then $\mathcal{D} \in \mathcal{F}$ since $\mathcal{D}$ is a collection of cycles, and if ϵ_1 and ϵ_2 are cycles in $\mathcal{D}$, then there is a $\mathcal{D}_{\alpha_0}$ which contains both ϵ_1 and ϵ_2, since $\mathcal{D}_\alpha$ is totally ordered. Since the cycles in $\mathcal{D}_{\alpha_0}$ are disjoint, the cycles ϵ_1 and ϵ_2 are disjoint, and so $\mathcal{D}$ is an upper bound of the chain.

This means that every totally ordered subfamily of $\mathcal{F}$ has an upper bound, so by Zorn's lemma, there is a maximal element $\mathcal{M}$ in $\mathcal{F}$. We now claim that the union of the cycles of $\mathcal{M}$ is the entire basis of the Hilbert space. Indeed, if e was a basis element that was not in $\cup\mathcal{M}$, then by Lemma 3.1.3, e is in some cycle, and by the same lemma, this cycle is disjoint from all the cycles in $\mathcal{M}$. Thus, $\mathcal{M} \cup \{\text{the cycle of } e\}$ would be a bigger cycle than $\mathcal{M}$, contradicting the maximality of $\mathcal{M}$.

Since $\cup\mathcal{M}$ is the entire basis, and since by Lemma 3.1.2 the subspace span of each cycle is a reducing subspace of A, we have that A decomposes into a direct sum of cycle parts of A. Moreover, by Lemma 3.1.1, these cycle parts are either finite cycles, unilateral shifts, or bilateral shifts. □

3.1.2 Permutations

From the previous discussion at the beginning of this chapter, we know that if A is a finite permutation matrix that is a semigroup matrix, then it must have (at the very least) that the orders of the disjoint cycles of A divide the order of some cycle of the permutation, or equivalently, A has a cycle whose order is the order of A. Examples of this would include the permutation consisting of a 3 and a 6 cycle, or a 2, 3, and a 6 cycle.

We first prove a lemma regarding associativity of binary operations.

Lemma 3.1.4. *A binary operation $*$ on a set S, defined by a collection of functions $L_x : S \to S$, $L_x(y) = x*y$, is a semigroup if and only if, for every $x, y \in S$, $L_x \circ L_y = L_{x*y}$.*

Proof. All that need be shown is that the binary operation is associative. For any $x, y, z \in S$, we note that $(x*y)*z = L_{x*y}(z)$ while $x*(y*z) = L_x \circ (L_y(z))$. So, $x*(y*z) = (x*y)*z$ for all $x, y, z \in S$, if and only if $L_x \circ (L_y(z)) = L_{x*y}(z)$ for all $x, y, z \in S$, thus, S is a semigroup if and only if $L_x \circ L_y = L_{x*y}$ for all x, y. □

For an easy construction giving a non-commutative semigroup now, we have the following:

Proposition 3.1.2. *Suppose A is an $n \times n$ permutation matrix. There is a cycle of A whose order is a multiple of all the other orders of cycles of A if and only if there is a semigroup S such that A is L_a for some a in S.*

Proof. It should be apparent that if A has a bunch of cycles, then the least common multiple of the orders of the cycles will be the order of A, and so it is obvious that if A arises as L_a of some semigroup, then a must be on a cycle of A whose order is a multiple of all the orders of the cycles of A.

For the converse direction, we will construct a semigroup such that $A = L_a$ for some $a \in S$. To do this, let the Hilbert space basis for A be denoted by S. We now want to define a multiplication on S. Pick any basis element a in a biggest cycle of the permutation, and fill in that row of the multiplication table with the permutation (so L_a will give us A on that cycle). We now have to fill in the rest of the table so that we get a semigroup.

First, we know that $a^2 = L_a(a) = A(a)$, and since associativity demands that $L_{a^2} = L_a L_a$, we must have $L_{a^2} = A^2$. Thus, multiplications by the powers of a are all determined by A, via $L_{a^k} = A^k$, with $a^k = A^{k-1}(a)$. The possible multiplications so far are well-defined and associative since for the order l of the cycle a is in, $A^l = I$ (since the orders of all the other cycles divide the order of the largest cycle, the order of the permutation is just the order of the largest cycle). Since $A^l = I$, we have that a^l is an identity (on what is defined so far), and for $i, j \leq l$, with $i + j \leq l$, $L_{a^i} L_{a^j} = A^i A^j = A^{i+j} = L_{a^{i+j}}$, while if $i + j > l$, $A^{i+j} = A^{i+j-l} = L_{a^{i+j-l}} = L_{a^{i+j}}$.

For the rest of the multiplications of the semigroup, define $xy = x$, where $x, y \in S$ with x not a power of a. An example of this construction with A having a 4 cycle and a 2 cycle is given below.

To show that the rest of this multiplication is associative, we just have to show that $L_x L_y = L_{xy}$. If both x and y are powers of a, then we have associativity from the above argument. If x is not a power of a, then L_x will send every basis element to x, so $L_x L_y(z) = L_x(yz) = x \Rightarrow L_x L_y = L_x$. Also, $L_{xy} = L_x$ follows from the definition of left multiplication by x. Finally, if x is a power of a but y is not, then $L_x L_y(z) = L_x(y) = xy$, so $L_x L_y$ sends every element to xy. Since xy is not a power of a (y is in a different cycle of the permutation than a, so $A^k y$ will never be in the cycle of a), then L_{xy} is also the basis map operator that send every element to xy, and so $L_x L_y = L_{xy}$. Thus, the constructed binary operation is associative and S is a semigroup with $L_a = A$. Moreover, since $A^l = I$ (for some $l \in \mathbf{N}$), $L_{a^l} = I$, and so a^l is a left identity. Since $a^k a^l = a^k$, and $xa^l = x$ for x not in the cycle of a, a is also a right identity, and so S is in fact, a unital semigroup.

□

$\cdot$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	d	a	f	e
c	c	d	a	b	e	f
d	d	a	b	c	f	e
e	e	e	e	e	e	e
f	f	f	f	f	f	f

Example of Proposition 3.1.2's construction with a 4 cycle and a 2 cycle as L_b.

It is understandable if one finds this construction rather uninspiring since the rest of the semigroup is just filled with idempotents that send everything to themselves. To get something more interesting, one may want a construction that gives a commutative semigroup instead. However, this case requires a slightly stronger assumption about the structure of A.

Proposition 3.1.3. *Suppose A is an $n \times n$ permutation matrix. There is a cycle whose order divides the orders of all the other cycles of A and there is a cycle whose order is a multiple of all the other orders of cycles of A if and only if there is a commutative semigroup S such that A is L_a for some $a \in S$.*

Proof. Suppose $A = L_a$. Then from Proposition 3.1.2 we know that A is in a largest cycle. Let g be an element in some cycle of order p and i some element of order m. By commutativity, $a^p(ig)$ must equal $i(a^p g) = ig$, but $a^m(ig) = (a^m i)g = ig$ as well, so ig is in a cycle of A whose order divides both m and p. Since this is true for any two elements g and i, and since there are only finitely many elements in the semigroup S, we must have that the order of a smallest cycle divides the orders of all the other cycles.

Now we proceed with the forward direction and construct a semigroup that A is obtained from. We identify the semigroup elements with the basis for A, leaving the construction down to defining the multiplication on these elements. For convenience, we will call the set of basis elements S.

From the arguments preceding this proposition, it is clear that A must be left multiplication by some element a in a largest cycle of the permutation, so we pick any element in one of these cycles (calling this element a) and define the left multiplication of a on S by their images under A, and extend this multiplication to all other powers a^i via $A^i = L_{a^i} = (L_a)^i$ and so on, as was done in Proposition 3.1.2. As before, we now have a (where composable) associative multiplication defined on the entire rows of the powers of a, and in the case of a single cycle permutation we obtain a cyclic group.

If A has more than more cycle then we now wish to extend multiplication further to the other cycles. Since the left multiplications of the powers of a are defined on all the semigroup elements, we can extend multiplication on S so that right multiplication by the powers of A are also defined, and are equal to the left multiplications. The composable products here also associate since all the left multiplications commute with the right multiplications. To extend the multiplication to the rest of the table, we pick a single element from each of the other cycles, which will become the generators of the semigroup along with a. Let m be one of these elements from a smallest cycle. For all pairs i, j of generators (neither of which are a), we define ij as $ij = am$, with $(a^k i)(a^l j) = a^{k+l}(ij) = a^{k+l+1}m$ for the other elements. By symmetry, this gives a commutative multiplication table, and since every element can be written as a power of a times a generator, this multiplication is also closed.

To see that it is associative, we first note that for a product $x(yz)$, if two of the elements are powers of a, then it associates since it's in the commutative extension of the a-groupoid, which was already shown to be associative. If only one of the elements is a power of a, then by commutativity we may assume that it's the x. Next, we rewrite the product in terms of the generators $x(yz) = a^k((a^l i)(a^p j)) = a^k(a^{l+p}ij) = a^k(a^{l+p+1}m) = a^{k+l+p+1}m$, while $(xy)z = (a^k a^l i)(a^p j) = (a^{k+l}i)(a^p j)$. Here however, we have to use the fact that the order of the cycle that m is on divides the order of all the other cycles. This is because $k+l$ may be bigger than the order of the cycle that i is on so $a^{k+l}i = a^{k+l \bmod \lambda_i}$, where λ_i is the order of the cycle that i is on. Thus, $(a^{k+l}i)(a^p j) = a^{(k+l \bmod \lambda_i)+p+1}m$, and in order for this to equal the above, it must be equal modulo the order of the m cycle (which we denote λ_m), so in other words,

$$(k + l + c\lambda_i) + p + 1 \equiv k + l + p + 1 \bmod \lambda_m \Rightarrow c\lambda_i \equiv 0 \bmod \lambda_m.$$

Since λ_m divides λ_i, this is true, and so both products are indeed equal[2]. Likewise, if none

[2]As implied earlier though, we must have that λ_i is actually a multiple of λ_m. For example, if we took the product $(a^p a^k i)(a^l i)$ instead, then c would be either 0 or -1 (since k and l are less than λ_i). Choosing k and l such that $k + l > \lambda_i$ will make this c equal -1, showing that λ_i must be a multiple of λ_m.

of the elements are a power of a, then we get a similar result of both sides as $a^{k+l+p+2}m$.

As an example to illustrate the process, the multiplication table for the 8-4-2 cycle permutation is given after this proof. □

As noted before, while we may be dealing with permutation (hence invertible) matrices here, the semigroups involved will never be groups themselves. It is easy to show that a basis map operator coming from a left multiplication in a group must be a permutation with all the cycles the same length. As well, given such a basis map operator, one can obtain a group with it as a left multiplication by just taking the direct sum of two cyclic groups (for example, the permutation consisting of two 3 cycles can arise from the group $Z_3 \oplus Z_2$ as multiplication by the generator of the 3 cycle).

$\cdot$	a	b	c	d	e	f	g	h	i	j	k	l	m	n
a	b	c	d	e	f	g	h	a	j	k	l	i	n	m
b	c	d	e	f	g	h	a	b	k	l	i	j	m	n
c	d	e	f	g	h	a	b	c	l	i	j	k	n	m
d	e	f	g	h	a	b	c	d	i	j	k	l	m	n
e	f	g	h	a	b	c	d	e	j	k	l	i	n	m
f	g	h	a	b	c	d	e	f	k	l	i	j	m	n
g	h	a	b	c	d	e	f	g	l	i	j	k	n	m
h	a	b	c	d	e	f	g	h	i	j	k	l	m	n
i	j	k	l	i	j	k	l	i	n	m	n	m	n	m
j	k	l	i	j	k	l	i	j	m	n	m	n	m	n
k	l	i	j	k	l	i	j	k	n	m	n	m	n	m
l	i	j	k	l	i	j	k	l	m	n	m	n	m	n
m	n	m	n	m	n	m	n	m	n	m	n	m	n	m
n	m	n	m	n	m	n	m	n	m	n	m	n	m	n

Example of Proposition 3.1.3's constuction with a 8-4-2 cycle.

3.1.3 Infinite Cancellative Basis Map Operators

The above ideas of how to construct a semigroup from a permutation also generalize to the infinite dimensional case. As noted before, the cycle of a basis element will be the set of all images and preimages of that point under powers of A, where we assume A is injective. Thus, a cycle can be a subset of the basis on which A acts like a finite cycle, a bilateral shift (a $\mathbf{Z}$ cycle), or a unilateral shift (an $\mathbf{N}$ cycle). Also, we will make the convention that any finite cycle "divides" a $\mathbf{Z}$ cycle and also an $\mathbf{N}$ cycle, while $\mathbf{Z}$ cycles "divide" $\mathbf{N}$ cycles[3].

Proposition 3.1.4. *Let A be a cancellative basis map operator. Then A is a direct sum of cycles, with the orders of the cycles having a common multiple (and a common divisor) if and only if A arises as left multiplication in a unital, (commutative) semigroup. Moreover if A has an $\mathbf{N}$ cycle, then in the semigroup that A comes from, the a such that $A = L_a$ must be the second element in an $\mathbf{N}$ cycle. In the case that A arises as a left multiplication in a non-unital semigroup and A has an $\mathbf{N}$ cycle, the a for which $A = L_a$ can be either the first or second element in some $\mathbf{N}$ cycle.*

Proof. The fact that A is a direct sum of cycles comes from Theorem 3.1.1. The first case to consider is when all cycles are finite. As before, associativity demands the common multiple structure on the lengths of the cycles, and commutativity will require the common divisor structure on the lengths of the cycles. The construction given in the finite case, when allowed to extend to an infinite number of finite cycles, will give a semigroup from which A will arise, and likewise the commutative construction will give a commutative one. This construction holds since for any three product of x, y, and z, the products $x(yz)$ and $(xy)z$ exist inside a finite submagma[4], whose multiplication is exactly the multiplication obtained from the constructions of Propositions 3.1.2 and 3.1.3 (without a dividing cycle and with one respectively), and so, is associative.

For the case where there are some infinite cycles, we first note that if A has both a $\mathbf{Z}$ and an $\mathbf{N}$ cycle, then the multiplication must come from the latter cycle. To see this, call the element that gives A under its left multiplication, a. If a was in the $\mathbf{Z}$ cycle, consider the preimage of a under A, and call it b. Then $ab = a$ so $a(bx) = ax$ for all x in the basis, so, by left cancellation, b is a left identity. Now consider the preimage of b and call it c. Then $acx = (ac)x = bx = x$, so c is a right inverse of a, but a can't have a right inverse

[3]We note that this convention is purely for descriptive purposes.
[4]We recall that a magma is just a set with a binary operation that is closed and totally defined.

if there is an **N** cycle since multiplication by a will not be onto since the first element of the **N** cycle will not be in the range of left multiplication by a. This shows us that not only must the multiplication arise from the **N** cycle, but also that it must arise from either the lowest element in that cycle (the element without a preimage under A) or the next one (in which case the lowest element will be like a left identity). To construct the semigroup, we take the construction given in the finite case and apply it to the cycles given here, with **N** cycles being treated as the biggest. Thus, if there is an **N** cycle, a is taken as the second basis element (with the starting subsemigroup as the non-negative integers instead of a finite cyclic group), while in the case that the only infinite cycles are **Z** cycles, a is some element on some **Z** cycle (with the starting subsemigroup as the integers), and in both cases, A becomes L_a. Finally, if the commutative construction is applied then the result is a commutative semigroup.

Finally, we wish to show the reverse implication in the case that A has at least one infinite cycle and is an L_a for some $a \in S$ (the case of all cycles being finite was covered at the beginning). For the noncommutative case, we only require that A has a cycle whose order is a least common multiple of the others, which is covered by A having an infinite cycle (either A has an **N** cycle which we consider as a multiple of a **Z** cycle, or it only has **Z** cycles which we consider as a multiple of all finite cycles). For the case where S is commutative we need to show that the cycles have a common divisor. If A has only infinite cycles, then the order of the cycles have a common divisor (either they're all **N** or there's a **Z** cycle which is considered a "divisor" of an **N** cycle). If A has a finite cycle, then for any elements c, d on finite cycles of length m, n, then $a^m(cd) = (a^m c)d = cd$ and $a^n(cd) = c(a^n d) = cd$. Thus, orders of the finite cycles must have a common divisor, and since finite cycles "divide" infinite cycles, then all cycles have a common divisor. □

This construction however (like in the finite case) is rather uninspiring for the infinite case since one will notice that most of the time there will be some left multiplications that will be unbounded. So we will now focus on when the semigroup we get has only bounded operators.

Proposition 3.1.5. *Let A be a cancellative basis map operator over an infinite dimensional Hilbert space. If A arises as an L_a from a semigroup whose multiplications are all bounded (both left and right), and A contains a cycle of infinite length, then A does not contain any finite cycles.*

Proof. Let A be given by left multiplication by a and suppose A has a cycle of finite length (call this length m), with c in this cycle. Since a must be in an infinite cycle, a^n is distinct for all $n \in \mathbf{N}$. However, $a^m c = c$, and so $(a^m)^n c = c$ for all $n \in \mathbf{N}$, and so right multiplication by c sends infinitely many elements to c and so R_c is unbounded. □

This proposition now splits up the remaining results into two cases: the purely finite basis map operator and the purely infinite basis map operator.

Proposition 3.1.6. *Let A be a cancellative basis map operator over an infinite dimensional separable Hilbert space that does not have any finite cycles. A comes from a commutative semigroup whose multiplications are all bounded if and only if it is not the case that A has both a finite number of $\mathbf{Z}$ cycles and an infinite number of $\mathbf{N}$ cycles.*

Proof. For the forward direction, assume otherwise that A has finitely many $\mathbf{Z}$ cycles and infinitely many $\mathbf{N}$ cycles. From Proposition 3.1.4, if $A = L_a$ then a is in an $\mathbf{N}$ cycle. Let c be any element in an $\mathbf{N}$ cycle, and f an element in the $\mathbf{Z}$ cycle. The product fc must be in an $\mathbf{Z}$ cycle. To see this, suppose that fc was an element in some $\mathbf{N}$ cycle and call it d. Let n be a natural number such that d has no n-th preimage under L_a (that is, there is no y such that $a^n y = d$). Then let x be in f's cycle such that $a^n x = f$ (this is possible since f is in a $\mathbf{Z}$ cycle). We now have $a^n xc = fc = d$, so xc is an n-th preimage of d under L_a, a contradiction, to the assumption that fc was in an $\mathbf{N}$ cycle. Thus, anything in a $\mathbf{Z}$ cycle times anything in an $\mathbf{N}$ cycle must be in an $\mathbf{Z}$ cycle.

Continuing, we let f be any element in one of the $\mathbf{Z}$ cycles (of which there are only finitely many). Let $\{c_i\}_{i \in \mathbf{N}}$ be a subset of elements from the $\mathbf{N}$ cycles such that there is exactly one c_i for each $\mathbf{N}$. Since there are infinitely many $\mathbf{N}$ cycles and finitely many $\mathbf{Z}$ cycles, there must be a subset c_j of the c_i's such that $\{fc_j\}$ are all in a particular $\mathbf{Z}$ cycle, and $\{fc_j\}$ is an infinite set.

This however means that L_f is unbounded. For any $k \in \mathbf{N}$, we can take the first k elements in $\{fc_j\}$, name the lowest one on the $\mathbf{Z}$ cycle d, and the rest $a^{l_0}d$, $a^{l_1}d$, ..., $a^{l_{k-1}}d$. Letting the biggest power be l_p, we have that $\{a^{i_j}fc_j\} = a^p d$ for some powers i_j. Commuting a^{i_j} through, we get that L_f sends k elements to $a^p d$, and moreover, these k elements are distinct since the c_j's were chosen to be in separate components (thus $a^m c_0$ can never equal $a^n c_1$). So for any $k \in \mathbf{N}$, we can find an element that appears k times in the f row of the multiplication table. Thus, L_f is unbounded, contradicting our assumption that A had finitely many $\mathbf{Z}$ cycles and infinitely many $\mathbf{N}$ cycles.

For the reverse direction, we start by assuming that there are finitely many $\mathbf{N}$ cycles, and call this number k. Let a_i be the lowest elements in the $\mathbf{N}$ cycles, and let $\{c_j\}$ be a collection of exactly one element from each $\mathbf{Z}$ cycle (this collection may be empty, a finite set, or infinite). We define multiplication by a_0 as the identity, and $a_j a_i = a_{i+j}$ if $i + j < k$, $a_j a_i = c_{i+j-k}$ if $i + j \geq k$, $a_j c_i = c_{i+j}$, and $c_i c_j = c_{i+j}$. If there are only finitely many $\mathbf{Z}$ cycles (say l), then we make the multiplication loop back to c_0 (that is, we have the relation that $c_l = c_0$). We let a denote the image of a_0 under A, and we call the element preceding c_0, a^{-1} (that is, $aa^{-1} = c_0$). All the elements of the basis of H can now be written as the pairs $\{(a^i, x) \mid i \in \mathbf{Z}, x = a_0, a_1, ..., a_k, c_0, c_1, ...,$ and $i \geq 0$ if $x = a_0, a_1, ..., a_k\}$, and multiplication on the semigroup will be multiplication of the pairs, with the understanding that a^i acts like the integers and the elements $a_0, ..., a_k, c_0, ...$ act either like $\mathbf{N}$ (if there are infinitely many $\mathbf{Z}$ cycles), or if there are l $\mathbf{Z}$ cycles, then the singly generated semigroup with the relation $c^{k+l} = c^k$.

We note that the above multiplication is well-defined, commutative, and associative since it is a subsemigroup of either $\mathbf{Z} \oplus \mathbf{N}$ (if there are infinitely many $\mathbf{Z}$ cycles) or $\mathbf{Z} \oplus C$ (if there are finitely many), where $\mathbf{Z}$ is the integers, $\mathbf{N}$ the non-negative integers, and C is the singly generated semigroup with the relation $c^{k+l} = c^k$. However, it must be verified that this subsemigroup is actually closed. The only issue of closure is if we have $(a^{-i}, {a_1}^l)$, where $l < k$, since these elements are not in the described subsemigroup. However, in the subsemigroup we have described, we can only get a negative power of a if the power of a_1 is greater than or equal to k, and by the construction, once we have this, multiplying ${a_1}^k$ by any power of a_1 will always give a power of a_1 bigger than k. Thus, multiplication here is indeed closed.

All the left multiplications in this semigroup have a cardinality norm at most k due to the fact that they are all injective everywhere except for the elements that fail to be injective in the second coordinate (the powers of a_1), of which there are k. Finally, by construction, A is given by $L_{(a,e)} = L_{(a,a_0)}$.

The other two cases we have to consider now is when A has no $\mathbf{N}$ cycles, and when it has both infinitely many $\mathbf{N}$ cycles and infinitely many $\mathbf{Z}$ cycles. For the first case, we can just write A as either $L_{(1,0)}$ in $\mathbf{Z} \oplus \mathbf{Z}$ if there are infinitely many $\mathbf{Z}$ cycles, or as $L_{(1,0)}$ in $\mathbf{Z} \oplus \mathbf{Z}/k\mathbf{Z}$ if there are k $\mathbf{Z}$ cycles. For the second case we can do a similar construction to the one above by first taking the semigroup $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{N}_2$, where $\mathbf{N}_2$ is the semigroup with 0 and 1 with multiplication as the maximum (by Theorem 2.1.1, this is a uniformly bounded semigroup). The constructed semigroup then becomes $\{(x, y, z) \in \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{N}_2 \mid y \geq 0 \text{ if } z = 0\}$. This is closed as before since if the product of two elements in this set were outside the set, then the product would have to have $y < 0$ and $z = 0$. This means that both factors in the product would have to have their z values equaling 0, but then both of their y values would have to be non-negative, giving a product whose y value is non-negative, and hence, an element in the set. Likewise, multiplication is commutative, associative, and bounded since these are preserved under taking direct sums and subsemigroups, and our matrix A comes as $L_{(0,1,0)}$.

□

A similar situation also happens in the finite cycle case if we wish to maintain boundedness of the semigroup.

Proposition 3.1.7. *Let A be a cancellative basis map operator over an infinite dimensional separable Hilbert space, with A having only finite cycles. Then A is a L_a for some a in a commutative, bounded semigroup if and only if A has a cycle of maximum length which is a multiple of the lengths of all the cycles and A has a smallest cycle which divides the order of all others, and there are infinitely many cycles of this order.*

Proof. By the same argument as before, if A is L_a for some a in a semigroup S, we must have that a is on the largest cycle, and that the order of the a cycle is a multiple of all the others. Thus, we need only worry about the second statement concerning the smallest cycle and it appearing infinitely many times.

We prove the forward direction first. Proceeding by contradiction, we assume that the smallest cycle only occurs finitely many times. Let c be any element of a smallest cycle. Then any element outside of these smallest cycles when multiplied by c will have to be in one of the smallest cycles. Since there are infinitely many cycles outside of the smallest cycle, we have that L_c sends an infinite set to a finite set, thus it must send an infinite subset of this infinite set to a single element. Hence, L_c is unbounded.

Finally, all we need to show is that if there are infinitely many cycles of the smallest order, then we can put A inside a commutative, bounded semigroup. For this, we consider some special cases first and then generalize to include all possibilities. First, if every cycle of A occurs infinitely many times, we can do a very simple construction: Let T be the semigroup obtained by the construction given in the finite case, but applied to only a single copy of each cycle (since there has to a cycle of maximum length, this semigroup will indeed be finite), and then take $S = T \oplus \mathbf{N} \cup \{\mathbf{0}\}$. In this semigroup, A is just $L_{(a,0)}$, where a is the multiplication of one of the elements of the largest cycle in T.

Now we consider the case where only the smallest cycle occurs infinitely many times. Here, we take the same semigroup S as constructed above, but we add on an additional summand of N_m, where N_m is the first m nonnegative integers with the maximum as the multiplication. For this particular construction, we will take m as the maximum number of cycles of any particular cycle (so for example, if we had two 4-cycles and three 2-cycles, we would take $m = 3$). Our multiplications now contain too many cycles, but the N_m semigroup allows us to take some interesting subsemigroups. We take the subsemigroup corresponding to the elements of the form (b, i, k), where $i > 0$ only if b is in the smallest cycle in T (this is ok since any smallest cycle times another smallest cycle will again be in a smallest cycle). Right now we have that $L_{(a,0,0)}$ is a basis map operator with infinitely many smallest cycles, and m of every other type of cycle. Next, we take another subsemigroup by removing any extra biggest cycles with the larger values of N_m. So if m was 4, but we only wanted two largest cycles, we would remove the elements of the form $(a, 0, 4)$ and $(a, 0, 3)$, where a is in the largest cycle. Likewise, to remove other finite cycles, we do the same thing, but remove their lowest values of m (so in the above example with a as a cycle that is neither largest or smallest, we would remove $(a, 0, 0)$ and $(a, 0, 1)$. Since a cycle times another cycle will be a cycle whose order divides the two, and due to the selection of the N_m parts of the above[5], this multiplication is closed, and so forms a subsemigroup, moreover, with $L_{(a,0,0)}$ as A again.

Now we have to consider the cases where some of the cycles that are not the smallest cycle are infinite, while others are finite. If there are not an infinite number of the largest cycle, then we can just take the above semigroup, and include the elements (b, j, i), b in the corresponding cycle in T, $i \in N_m$, $j \in \mathbf{N}$. Since in the semigroup T, b times any element that is not in the largest cycle is in the smallest cycle, and that b times any element in the largest cycles is just in the b cycle, we have that this bigger set is also a subsemigroup, and that $L_{(a,0,0)}$ is the A in this case. Finally, we have to consider what happens when there are infinitely many cycles of the largest type, and some other cycles

[5]It is important here that the largest cycle have the higher N_m values removed while the smaller cycles have the lower N_m removed. Otherwise, under the multiplication, we could have something in a largest cycle with a high N_m coordinate times a something in a smaller cycle with a low N_m. This would yield something in the smaller cycle, but with a high N_m coordinate.

that only have finitely many. Here, we have to change the beginning semigroup T to be the one constructed in the finite case, but with two copies of the largest cycle given. In this new semigroup T' we note that the other largest cycle will send everything into the smallest cycle. Thus, to obtain A, we take apply the same idea as in the above case, but treat the other largest cycle as a different cycle from the largest cycle (so we can add the elements (a', i, j), where a' is the other largest cycle, $i \in \mathbf{N}$, and $j \in N_m$). So, in this case we get infinitely many largest cycles, finitely or infinitely cycles of the other intermediate sizes, and infinitely many of the smallest size, making $L_{(a,0,0)}$ again the multiplication that gives A, which finishes the proof. □

The constructions given in the above may seem little bit strange since the desired semigroup is a particular subsemigroup of some direct sum of semigroups. While this may seem an odd way of constructing the semigroup, it should be noted that the common method of decomposing commutative semigroups is to consider them as subsemigroups of direct products of irreducible semigroups. While the above constructed factors are rarely irreducible, the same idea still holds.

We end this section by summarizing the results so far.

Theorem 3.1.2. *(Unbounded Semigroup Problem for Cancellative Basis Map Operators) Let A be a cancellative basis map operator, then*

1. *A can be realized as left multiplication in a semigroup if and only if there is a fixed cycle in which all the orders of the cycles of A divide the order of the fixed cycle.*

2. *A can be realized as left multiplication in a commutative semigroup if and only if there there is a fixed cycle in which all the orders of the cycles of A divide the order of this fixed cycle, and there is a fixed cycle whose order divides all the orders of the cycles of A.*

Theorem 3.1.3. *(Bounded Commutative Semigroup Problem for Cancellative Basis Map Operators) Let A be a cancellative basis map operator for an infinite, separable Hilbert space, then*

1. *if A only has cycles whose orders are finite, then A can be realized as left multiplication in a bounded commutative semigroup if and only if the orders of the cycles of A have a common multiple and a common divisor, with the common divisor cycle appearing infinitely many times.*

2. *if A has only infinite cycles, then A can be realized as left multiplication in a bounded commutative semigroup if and only if it is not the case that A has an infinite number of $\mathbf{N}$ cycles and a finite non-zero number of $\mathbf{Z}$ cycles.*

3. *if A has both finite and infinite cycles, then A can not be put into a bounded semigroup.*

Theorem 3.1.4. *(Partial Results for Bounded Semigroup Problem for Cancellative Basis Map Operators) Let A be a cancellative basis map operator for an infinite, separable Hilbert space, if A can be realized as left multiplication in a bounded semigroup, then*

1. *A has either all finite cycles or all infinite cycles.*

2. *if A has all finite cycles, then there must be a cycle whose order is a common multiple of all the others.*

We remark that the bounded semigroups constructed in this section were all semigroups with a countable number of elements. To get a similar version of Theorem 3.1.3 with a non-separable Hilbert space, one would have to require that the smallest cycle would have to occur at least as many times as any other cycle. For example, if the number of $\mathbf{N}$ cycles has a cardinality of κ, then the number of times that the smallest cycle occurs has to have a cardinality of at least κ. Moreover, since there are only a countable number of different types of cycles, this means that the number of times the smallest cycle occurs must have the same cardinality as the number of basis elements of the underlying Hilbert space.

3.2 Connected Semigroups

Now we wish to go on to the more general case when A is a basis map operator, but not injective. To understand what these semigroup operators can look like, we will first try to understand how two left multiplications can relate to each other inside a semigroup. That is, if we know what L_b looks like for some $b \in S$, what can we deduce about L_a for some other $a \in S$? To help with this, we will consider graphs associated to multiplications as well as some functors of what we will call pointed semigroups.

3.2.1 Multiplication Graphs and Connectedness

Our first step to understand the structure of L_a, $a \in S$, will be to consider it visually as a directed graph. Since semigroup matrices have a very special form, such a visualization comes quite easily.

Definition 3.2.1. *For a semigroup operator A (or more generally, a basis map operator), its* **multiplication graph** *(or sometimes just referred to as its* **graph***), will be the directed graph that results from the matrix of the operator when one considers it as an adjacency matrix.*

Specifically, the graph is what one gets when one labels the orthogonal basis of A as vertices, and draws an edge going from e_i to e_j if $A(e_i) = e_j$.

It should be noted that this description of a basis map operator does not give any additional information. However, this visualization will prove to be useful in describing the properties of semigroup matrices.

As an example of such a multiplication graph, consider the three element semigroup with elements A, B and C, with multiplication defined as the min operation with $A <$

$B < C$. The adjacency graph of L_A would be as in figure 3.1 below.

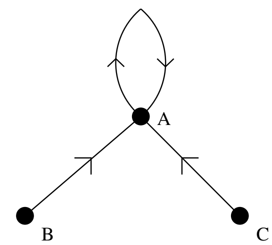

Figure 3.1: The directed graph of L_A

The first definition that we'll need for this section will be when two elements of the graph of A are connected as an undirected graph, that is, connected when one is also allowed to go in the opposite direction of the arcs as well. A more useful definition for this concept is the algebraic description instead.

Definition 3.2.2. *For L_a, $a \in S$, and $x, y \in S$, we will say that x is* **connected** *to y (via a) if there are $n, m \in \mathbf{N}$ such that $a^n x = a^m y$.*

We first note that two elements x and y in S which are connected to each other in the above sense, if and only if the corresponding vertices of x and y in the undirected graph of L_a are connected to each other. Indeed, if x is connected to y in the undirected graph, then a path starting from x and ending at y is a finite sequence of edges to choose to follow. These choices are either in the direction of the edge (of which, by definition of the graph of L_a, this is a unique choice for each vertex), or in the opposite direction of the edge. Since going in the opposite direction of an edge and then going in the direction of the next edge acts as the identity (again, since the choice of the latter is unique), the existence of a path from x to y is equivalent to the existence of a path that goes in the direction of the edges for some amount of steps and then in the opposite direction for some other number of steps. The vertex of the graph where the path switches from going in the direction of the edges to the opposite is given algebraically by $a^n x$ and as $a^m y$, where n is the number of times the path goes in the direction of the edges, and

m the number of times it does in the opposite direction. Likewise, if one is given that $a^n x = a^m y$, then $a^n x$ and $a^m y$ define paths going from x to the vertex d and y to d, so the composition of the first, with the reverse of the second gives a path on the undirected graph of L_a going from x to y.

We also note that this definition of connectedness gives us an equivalence relation on the elements of S, whose equivalence classes are (due to the equivalence of the definitions given above) the connected components of the graph of L_a.

We remark that in the case of a group, then the connected components of L_a correspond to orbits of elements under multiplication by a, or equivalently, cosets of the group by the cyclic subgroup generated by a. In particular, the connected component that a belongs to corresponds to the subgroup generated by a. In the semigroup case, this connected component may only contain the subsemigroup generated by a, but it is still a subsemigroup (though not necessarily a commutative one).

Proposition 3.2.1. *(Connected Subsemigroup Generated by a) For $a \in S$, the connected component of L_a containing a is itself a subsemigroup of S.*

Proof. We need only show that for any two elements x, y connected to a, their product is also connected to a. Since x is connected to a, by the algebraic definition of connectedness, there are $m, n \in \mathbf{N}$ such that $a^n x = a^m a$, and likewise there are $j, k \in \mathbf{N}$ such that $a^j y = a^k a$. Thus, $a^{n+j} xy = a^j (a^n x) y = a^m (a^j y) = a^{m+k} a$, so xy is also connected to a. □

Next, we note that in the general case of A being a finite matrix, but not being a permutation, then there will be a subset of the basis on which A does act like a permutation, and the subgraph corresponding to this subset will consist of disjoint cycles, which will give us something to build on, using the constructions for the cancellative case. In the infinite case however, the situation becomes a bit more complicated since one may have a unilateral shift or an infinite branch of elements going into a either a finite cycle or something that looks like that bilaterial shift.

One way to address this is to just define the permutation part of an infinite dimensional basis map operator A in the same way that one defines the unitary part of an isometry as just the intersection of its ranges (and specifically in our case, the intersection of the ranges of the subset corresponding to the orthogonal basis). This however will not be that beneficial since the goal of considering such a permutation part is to have something to build on for each connected component of the multiplication graph, and for a connected component that looks like the unilateral shift this definition will just give us the empty set.

Instead, the approach we will use is to consider a not necessarily unique permutation part of a basis map operator. Since "permutation part" could mislead one into thinking that the basis map operator acts surjectively as well, we will use the terminology of a "cycle part decomposition" instead. This will turn out to be more appropriate for future notation as well.

Definition 3.2.3. *For a basis map operator A whose graph is connected, we call a subset M of the basis maximally injective if A acts injectively on M, and for any subset N of the basis outside of M, the powers of A are either not injective or not closed on $M \cup N$. A maximally injective M will be said to be a* **cycle** *if for any element c outside of M, then there is an element $d \in M$ such that for some $n \in \mathbf{N}$, $A^n(d) = A^n(c)$. As well, $A|_M$ will be called a* **cycle part** *of A.*

A **cycle decomposition** *(respectively a* **cycle part decomposition***) with respect to a basis map operator graph will consist of a union of cycles, one for each connected component of the basis which has a cycle (respectively the direct sum of cycle parts for its connected components with cycle parts). Finally, if a basis map operator has a cycle for each connected component of its graph, then it will be said to have a* **full cycle decomposition***.*

For a connected A it not obvious that a cycle part should exist (nor for any connected component of a basis map operator), but if such a cycle part does exist, then by injectivity and connectedness it is either a finite cycle of order n, a bilaterial shift, or a unilateral shift. Moreover, if there is another cycle part, then it must be of the same type as listed above. If a cycle type is that of a finite cycle of order n, then the cycle itself is actually unique since any element in the connected component of the cycle must go into this cycle under enough powers of A, and by injectivity, this forces any other cycle of the component to just be this cycle itself.

If a cycle part of a component is the unilateral shift, then any other cycle part of the same component can not be the bilateral shift. If it was, then the first basis element of the unilateral shift (call it c) would be connected to some element (call it d) of the bilateral shift. This means that $A^n c = A^m d$, and since d is in a bilateral shift we can pick an n-th predecessor of it (call it f). Then $A^n c = A^{m+n} f$. This violates the last condition of a cycle. Indeed, if $A^j g = A^j f$ for some $j \in \mathbf{N}$ and g in the $\mathbf{N}$ cycle, then for some $i \geq j$, $A^i c = A^j f$...(1). However, we also know that there are $n, m \in \mathbf{N}$ such that $A^n c = A^{m+n} f$...(2) (ie: the power of A for c is less than the power of A for f). Multiplying both sides of equation (1) by a power of A to get the same power of A for the c element of equation (2) and then equating the right hand sides gives that $A^r f = A^s f$ where $r < n < s$. This means that the graph of A has a finite cycle in it, giving us a contradiction. Thus, if a cycle of a component is an $\mathbf{N}$ cycle, then any other must be a $\mathbf{N}$ cycle as well. Finally, by the above, if a component has a $\mathbf{Z}$ cycle, by the previous arguments, it can't have a finite cycle or $\mathbf{N}$ cycle and so any other cycle must be a $\mathbf{Z}$ cycle as well. Thus, if a component has a cycle, then any other cycle of the component must be of the same cycle type.

Formally speaking however, we can specify a better definition of cycle type.

Definition 3.2.4. *For a connected basis map operator A, define an equivalence relation on the basis of A by $x \equiv y$ if for some $n \in \mathbf{N}$, $A^n x = A^n y$ (note that n here is the same power for both x and y). The induced map of A on these equivalence classes, acts* *injectively and is connected, and so is either a finite cycle, the unilateral shift, or the bilateral shift. We say that A has* **cycle type** *of n if this induced map is a finite cycle of order n, of type* **N** *if it is the unilateral shift, and* **Z** *if it is the bilateral shift.*

Likewise, in the case of a basis map operator which is not connected, we can also use the above definition to define a cycle type for each connected component of the graph of the operator.

The induced map of A on the equivalence classes is well defined since if two elements are equivalent, then their images under A are in the same equivalence class. Likewise this induced map is also injective since if $Ax \equiv Ay$, then $A^n Ax = A^n Ay$ for some n, so $A^{n+1}x = A^{n+1}y$, and thus $x \equiv y$. Finally connectedness of the induced map follows from connectedness of A.[6]

While the cycle type for a connected basis map operator will always exist, and as shown earlier, will be the same for any choice of cycle, this does not mean that every such operator will have a cycle part. For example, take the unilateral shift, and then add on an arc of length two going into its second basis element (so we've added two basis elements, with one going into the shift's second element, and another going into this added basis element). Next add an arc of length 4 going into the third element, then an arc of length 6 going into the fourth element, and so on. Call this basis map operator B.

This constructed operator B has cycle type **Z** since each time we added an arc, we increased the number of cycle type equivalence classes by one, and if we keep adding them infinitely, then any equivalence class of the induced map of B will always have preimages. So if this operator had a cycle part, then B would act like the bilateral shift on some subset of the basis. However, by construction, every element of the basis has only a finite chain of predecessors under the map B (either the basis element is in an attached arc which has only finitely many, or it is on the unilateral shift part, which also has finitely many since the arcs kept being added further along the shift), so this is

[6]This induced map on equivalence classes will be studied later in section 3.2.3.

impossible.

Being of $\mathbf{Z}$ type and every basis element having only finite chains of predecessors classifies such connected basis map operators.

Proposition 3.2.2. *B is a connected basis map operator without a cycle part if and only if B is of $\mathbf{Z}$ cycle type and every basis element has only finite chains of predecessors under B.*

Proof. The reverse direction follows the same logic as the argument given for the constructed operator above. Thus we need only show the forward direction.

Suppose that B is a connected basis map operator without cycle part. If B is not of $\mathbf{Z}$ cycle type, then either it is of cycle type n or cycle type $\mathbf{N}$. If it is of cycle type n, then $x \equiv A^n x$, so for some $m \in \mathbf{N}$, we have that $A^m x = A^{m+n} x$, so the elements $A^m x, A^{m+1} x, ..., A^{m+n-1} x$ form an n cycle. This is the case since these elements form a maximally injective subset of the basis, and for any x outside of it, for some $k \in \mathbf{N}$, $A^k x$ will be inside the cycle, and going backwards in the cycle k times will give the desired cycle element y such that $A^k x = A^k y$, and thus, this component has a cycle.

If B is of cycle type $\mathbf{N}$, then the equivalence classes on which the induced map of B acts has a smallest element. Picking any basis element in this equivalence class (call it y), and then taking all powers of it gives a maximally injective subset of the basis. Likewise, for any x outside this set, then $A^n x = A^m y$, where $m \geq n$ by the minimality of the equivalence class that y belongs in. Then $A^n x = A^n (A^{m-n} y)$, so $A^{m-n} y$ is the desired element in the set, so B has a cycle part. Thus, B must be of $\mathbf{Z}$ cycle type.

Finally, we need only show that every basis element must have only a finite chain of predecessors under the map B. If it did not, then for some B there is an element y with an infinite chain of predecessors. Take the set M consisting of this element along with this infinite chain of predecessors, as well as all its successors under B. Then this set is a maximally injective subset of the basis on which B acts like the bilateral shift. Also, for any x outside of the set, we have that for some $m, n \in \mathbf{N}$, $A^m x = A^n y$. If $n > m$, then $A^m x = A^m A^{n-m} y$ with $A^{n-m} y \in M$, and if $n < m$, then $A^m x = A^m z$, where z is the $m - n$ predecessor of y in M. Thus M is a cycle of B, giving a contradiction. Thus every basis element must have only finite chains of predecessors under B. □

In some sense, such basis map operators are "between" being of **Z** type and being of **N** type, and cause a problem to the idea of starting with a given cycle type and adding on additional elements going into it. However, while such connected basis map operators exist, we will find out later that they can never exist as connected semigroup operators themselves, which help us extend the ideas of the previous section to non-cancellative, but connected basis map operators.

Due to the maximality of the cycles, for every component of a basis map operator that has a cycle part, every element outside the cycle will eventually go into the cycle under enough iterations of the operator. To help with the exposition of this in the future, we will call these elements the **hairs** of the graph.

Definition 3.2.5. *For a basis map operator A, an element d in a connected component with a cycle, but not in that cycle itself, will be called a* **hair** *for that cycle. A hair will be called a hair of length k if the minimum number of iterations of A required for d to go into the cycle is k. Finally, a set consisting of a hair of length l along with its images under powers of A that are not in the cycle, will be called a* **branch** *of hairs of length k.*

We note that any hair or branch of hairs of a basis map operator will have to be of some length (in the case of a branch, possibly infinite), due to the maximality of the cycle of the connected component it lies in.

As an example of hairs, we see the example of figure 3.1 above, both the elements B and C would be considered hairs (both of length 1), while A would be considered part of the cycle.

Finally, we end this section with a lemma about the connected components of semigroup operators from commutative semigroups.

Proposition 3.2.3. *Let L_a be a semigroup operator for an element a in a commutative semigroup S. If x is a-connected to y and c is a-connected to d, then xc is a-connected to yd. In other words, the connected components of L_a form a well-defined quotient semigroup of S under the connected equivalence relation.*

Proof. Letting x, y, c, and d be as described in the above, we have that $a^i x = a^j y$ and $a^m c = a^n d$ for some $i, j, m, n \in \mathbf{N}$. Multiplying these two equations together, we have that $a^i x a^m c = a^j y a^n d \Rightarrow a^{i+j} xy = a^{m+n} cd$, showing that xy is a-connected to cd. □

We remark that such a multiplication need not exist for non-commutative semigroups. Indeed, the non-commutative examples constructed previously do not have such a property.

3.2.2 Pointed Basis Map Operators

From our study of cancellative basis map operators, we have a good understanding of the possibilities for the cycle parts of a semigroup operator, so our goal now is to expand this to include the more general case and ask what kind of hairs a given cycle can have.

If we start with the graph of a cancellative, finite dimensional basis map operator, then we can add as many hairs of length 1 as we like to the permutation and still have a graph that can arise as some left multiplication in some semigroup. To see this, suppose that S is a semigroup with L_a corresponding to the starting permutation (like the construction used in Proposition 3.1.2). If one adds a hair to this cycle, then there is an element c in that cycle that will be sent to the same thing as the element that was just added. The semigroup can then be extended by just adding on an element to it and declaring that it acts just like c did, and from here we can do this as many times as we like, as long as the hairs are added to the permutation part and that we don't add infinitely many hairs to a single element or have an increasing, unbounded number of hairs added to a sequence of elements (as this would give an unbounded operator).

If we want a semigroup with an identity however, we must have the identity element going into a in the graph, and if the hairs are all of length 1 then a will must be in the cycle. Moreover, a must have a hair going into it since otherwise the identity will be on the cycle, meaning that L_a must be invertible (since $L_a^n = L_e = I$ for some $n \in \mathbf{N}$). So if we want a semigroup with an identity, then we can do the same construction as the above, except we must choose a to have at least one hair going into it, remove one of these hairs, form the semigroup as listed above from this modified graph, and then at the end adjoin an identity to this semigroup[7].

Increasing the possible lengths of these hairs beyond one vertex however is a little bit more difficult to do. Whereas before L_a corresponded to left multiplication by an element in the permutation, adding a second vertex to the hair fundamentally changes the structure of the semigroup. Take for example a simple n-cycle permutation with a branch of hairs of length 2 coming out of it like given in figure 3.2 below.

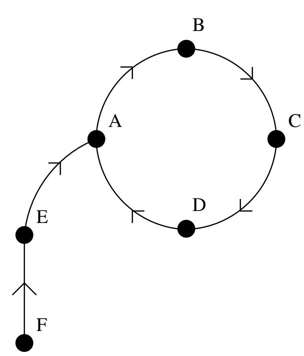

Figure 3.2: 4-cycle with F as a hair of length 2

Suppose that this graph corresponds to multiplication by some element a in the permutation[8]. We take the element c to be the elements that is two vertices behind

[7]In the case of a semigroup S which already has an identity e, we take the convention that "adjoining an identity" means that we add a new element e' to the semigroup to form a new semigroup S' with the property that $xe' = e'x = x$ for all $x \in S'$. In particular, since $ee' = e'e = e$, the identity of S is no longer the identity of S'.

[8]Here we use lowercase letters to indicate an arbitrary element a, while uppercase letters will refer the specific elements in the above graph.

a in the cycle of the graph, in particular making $L_a^2(c) = a$, so $L_{a^2c} = L_a$. Applying this operator to the second vertex in the hair F, and we get that $L_a(F) = L_{a^2c}(F) = L_{a^2}L_c(F) = (L_a)^2(cF)$. But, $L_a(F) = E$, so $E = (L_a)^2(cF)$, but this is impossible since E does not have a second predecessor in the graph, so it's impossible for there to be an element x such that $(L_a)^2(x) = E$. Moreover, by the same argument, the multiplication must come from either the last, or the second last vertex in the branch of hairs. If it comes from the last vertex, then the semigroup will not have an identity since in this case there is no element that is sent to a under L_a, so if the semigroup has an identity, then this multiplication must come from the second last vertex in the branch of hairs.

This discussion reminds us that semigroup operators have some additional information to them then just the graph or operator structure, namely, the semigroup operator has a distinguished element from which the multiplication arises. As such, one should not only classify which basis map operators are semigroup operators, but also which basis element the operator can arise from as a left multiplication in the semigroup. This gives us the idea of a basis map operator with a distinguished basis element, which we call a pointed basis map operator.

Definition 3.2.6. *The pair (A, a), consisting of a basis map operator A along with a basis element a, will be called a* **pointed basis map operator***. If A can be realized as a semigroup operator, with the multiplication coming from a in the basis, then the pair (A, a) will be called a* **pointed semigroup operator***.*

As seen in the above example, the distinguished element can't be freely chosen (unlike in the group case), and in fact we will see that the choice can be rather rigid. Before we look into this deeper however, we will investigate a natural grading that arises on the connected component of the element a in the graph of L_a.

3.2.3 The Functor G_a

We know that for a semigroup operator L_a, the connected component of an element is given in a nice algebraic way. Moreover, from the previous discussion on cycle type, if we look at the connected component of a, then we can form an equivalence relation by saying that two elements are equivalent if they are the same when we multiply by a large enough power of a. Due to connectedness, we can form a map from the elements of the connected component of a to the equivalence classes, each indexed by an integer.

Definition 3.2.7. *For a semigroup S with element $a \in S$ and M the connected component of a, we define the a-**grading functor** as the map $G_a : M \to \mathbf{Z}$ such that $G_a(x) = i$ if for some $n \in \mathbf{N}$, $a^n x = a^{n+i}$. In the case that the connected component of a has a finite cycle, we replace the target group $\mathbf{Z}$ with $\mathbf{Z}/m\mathbf{Z}$, where m is the order of the cycle.*

The interesting property of G_a is that it is both well defined, and with a suitable definition of morphisms of pointed semigroups, is also a functor from them to singly generated, cancellative semigroups ($\mathbf{Z}, \mathbf{N}$, and $\frac{\mathbf{Z}}{\mathbf{mZ}}$).

Lemma 3.2.1. *For a semigroup S, $a \in S$, and M the connected component of a, then G_a is well defined (the i such that $G_a(x) = i$ exists and is unique) and has the functoral property: if $x, y \in M$ then $G_a(xy) = G_a(x) + G_a(y)$, where addition is taken with respect to the cycle type of M (that is, either $\mathbf{N}$, $\mathbf{Z}$, or $\mathbf{Z}/m\mathbf{Z}$).*

Proof. The first thing to show is that the grading is well-defined, that is, for any $x \in M$, $G_a(x) = i$ for some $i \in \mathbf{Z}$, and that this i is unique with respect to the target group.

To do this, we note that since x is connected to a, $a^n x = a^k a$ for some $n, k \in \mathbf{N}$. However, we could have that $G_a(x) = i$ for two different $i \in \mathbf{N}$. Indeed, it could happen that for some $n \in \mathbf{N}$,$i, j \in \mathbf{Z}$, $a^n x = a^{n+i}$ and $a^n x = a^{n+j}$ (by taking the maximum of the two values of n and multiplying by an appropriate power of a, we can assume that both of the n's corresponding to i and j are the same). This gives that $a^n x = a^{n+i} = a^{n+j}$. Thus, the connected component of a must just have a finite cycle, and moreover, its order must divide $i - j$, and so i is congruent to j modulo the order of the cycle, and so $i \cong j$ in $\mathbf{Z}/m\mathbf{Z}$, making G_a well-defined.

Next, we wish to show that if $x, y \in M$, then $G_a(xy) = G_a(x) + G_a(y)$. Let $m, n \in \mathbf{N}$ such that $a^m x = a^{m+G_a(x)}$ and $a^n y = a^{n+G_a(y)}$. Then $a^{m+n}xy = a^n(a^m x)y = a^n a^{m+G_a(x)} y = a^{m+G_a(x)} a^n y = a^{m+n+G_a(x)+G_a(y)} = a^{m+n} a^{G_a(x)+G_a(y)}$. Thus, $G_a(xy) = G_a(x) + G_a(y)$. □

In semigroup theory language, the above lemma states that the equivalence classes consisting of elements with the same image under G_a form a congruence on the connected subsemigroup generated by a.

We know formally show that G_a is a functor from the category of connected semigroups[9] to the category of singly generated[10], cancellative semigroups.

Proposition 3.2.4. *For any connected semigroup S, with $a \in S$ such that the graph of L_a is connected, and any semigroup homomorphism ϕ from S to a semigroup T, then G_a commutes with ϕ in the sense that $\phi' \circ G_a = G_{\phi(a)} \circ \phi$, where ϕ' is the induced map of ϕ on the grading classes of G_a. Specifically, the map ϕ' is defined as the composition of the pullback map[11] to the cycle part of S, with the homomorphism ϕ, and then the grading map $G_{\phi(a)}$ on T.*

Proof. Let $b \in S$. Then we have to show that $\phi'(G_a(b)) = G_{\phi(a)}(\phi(b))$. Let the grading class of b be k, then $\phi'(G_a(b)) = G_{\phi(a)}(\phi(c))$, where c is the element in the cycle of S, which has a grading of k. So, if we can show that if two elements are in the same grading class under a in S, then their images under ϕ will be in the same grading class under $\phi(a)$ in T, then we'll have that $G_{\phi(a)}(\phi(c)) = G_{\phi(a)}(\phi(b))$, and we'll be done.

To show this, let's assume that g and h are in the same grading class under a in S.

[9] The objects of this category consist of connected semigroups, as defined previously. Morphisms between these objects are the usual semigroup homomorphisms.

[10] Here, singly generated means either singly generated as a semigroup, or as a group.

[11] For k in the grading class of G_a, this pullback map sends k to the element in cycle of S which has a grading of k.

From the definition of the grading, we have that for some $m \in \mathbf{Z}$, $a^m g = a^m h$, and thus $\phi(a^m)\phi(g) = \phi(a^m)\phi(h)$. This gives us that the grading classes of $\phi(g)$ and $\phi(h)$ are the same under $\phi(a)$, and so $\phi'(G_a(b)) = G_{\phi(a)}(\phi(b))$. □

We note that if the map ϕ is onto, then the induced map ϕ' between the singly generated semigroups will be onto. This is since the pullback map sends the singly generated semigroup to the cycle of S, and then the map ϕ has to send the 1-grading element of this cycle to a 1-grading element (under $\phi(a)$) in T (since $a^m a = a^m c$, implies that $\phi(a)^m\phi(a) = \phi(a)^m\phi(c)$). This element in the 1-grading class of T will then be able to fill in the remaining positive elements in the singly generated grading semigroup of T, while the fact that ϕ is onto guarantees that the negative grading classes must be mapped into by something in S (here we can pick something in a negative grading class of T, pick a preimage under ϕ, and then we have something in a negative grading class in S that will be mapped to a negative grading class in T).

We note that the onto requirement of ϕ is indeed needed here, since as a counterexample, we can take the homomorphism from $\mathbf{N}$ to $\mathbf{Z}$ under the inclusion map. Here, the grading maps are isomorphisms, and the induced map between the grading semigroups will just send $\mathbf{N}$ to $\mathbf{Z}$ like the inclusion map did.

3.2.4 Pointed, Connected, and Commutative Semigroup Operators

From our discussion before we were able to deduce that for a pointed semigroup operator L_a, the a can not be just any element. Here, we wish to classify pointed semigroup operators that are connected and arise from commutative semigroups. To start, we show that a connected semigroup operator must have a cycle part.

Proposition 3.2.5. *Let S be a semigroup, and $a \in S$ such that the graph of L_a is connected. Then L_a has a cycle part.*

Proof. By Proposition 3.2.2, we need only consider the case that the cycle type of L_a is $\mathbf{Z}$. From the grading we know that a lies inside the 1 grading class of G_a. Pick an element in the -1 grading class and call it c. Due to connectedness, we know that for some $k, l \in N$, we have that $a^l c = a^k$, moreover, from the grading function we know that we can choose k such that $a^{k+1}c = a^k$.

Now, for c^2, we know that $a^{k+1}c^2 = (a^{k+1}c)c = a^k c$. Likewise for c^3, $a^{k+1}c^3 = a^k c^2$, and in general, $a^{k+1}c^i = a^k c^{i-1}$. Next, we consider the set of elements $M = \{a^k c^i\}_{i \in \mathbf{N}}$. We have that $a(a^k c^i) = a^{k+1}c^i = a^k c^{i-1}$, so if we include the set $N = \{a^{k-1+i}\}_{i \in \mathbf{N}}$, then $M \cup N$ will contain exactly one element from every grading class of a, and a will act / injectively on $M \cup N$. Since it has an element in each grading class, this set is maximally injective, and for any y in the connected component of L_a, we can choose the element x in its grading class in $M \cup N$, to get that for some $j \in \mathbf{N}$, $a^j x = a^j y$. Thus $M \cup N$ is a cycle for the graph of L_a. □

Now that we know these graphs must have a cycle part, we try to understand the possible hair structure on these cycle parts and where the multiplication must arise from.

Lemma 3.2.2. *Let S be a unital, bounded, semigroup that has an element a for which the graph of L_a is connected. Then L_a either has a finite cycle, in which there are a finite number of hairs going into the cycle (hence S is finite) with the element a as one of the second furthest hairs away from the cycle (or an element on the cycle with a hair going into it if there are only hairs of length at most 1), or a has a infinite cycle of type $\mathbf{N}$ with each hair being of finite length, with a as a second element on the $\mathbf{N}$ ray, or a has an infinite cycle of type $\mathbf{Z}$ with a as the second last vertex of a finite hair or a vertex at an intersection of infinite branches if there are no branches of finite length.*

Proof. For the case that L_a has a finite cycle, suppose there is an infinite branch of hairs going into the cycle. For some number k, a^k has to be in the cycle, thus, $a^{k+m} = a^k$, where m is the order of the cycle. Let y be an element on the infinite branch that is more than k elements away from the cycle. Then $a^{k+m}y = a^k y$, but these two can not be equal since $a^k y$ is not yet in the cycle and $a^{k+m}y$ is further along the branch. Thus, if L_a's cycle is a finite cycle, then it must have hairs of length at most a fixed number. Likewise, a must be an element whose distance away from the cycle is the second largest, or if all the hairs are of length at most 1, then a is on the cycle with a hair going into it.

For the case of an infinite cycle of type $\mathbf{N}$, then a can't have a hair of infinite length, since then it would be of type $\mathbf{Z}$, so all hairs are of finite length. Moreover by the grading on S via a, a must be one of the elements that is second furthest back away from the $\mathbf{N}$ ray, otherwise there would be an element in the grading that corresponds to -1 and so it along with its powers would give an $\mathbf{N}$ cycle.

Finally, in the case of a $\mathbf{Z}$ cycle, by earlier remarks if there is a finite hair then a must be the second last vertex in one of the finite branches of hairs. The last thing we must show is that if L_a has only infinite hairs, then a must be at one of the intersection points of the infinite branches. Suppose otherwise, then a has only one preimage under L_a. Since S has an identity e, we know that this preimage must be exactly e. Now let g be any preimage of e. Then we know that $ag = e$, so a has a right inverse, but as well $a(ga) = (ag)a = ea = a$, so ga must be a preimage of a. Since a only has one preimage, we have that $ga = e$, so g is also a left inverse of a as well, so a must be invertible, which contradicts the fact that L_a is not injective. Thus, a must come from an intersection point on the graph if there are infinite branches and no finite ones. □

The last case of a $\mathbf{Z}$ cycle in the above is somewhat uninspiring and leads to the least restriction on the graph of a. However, if we know that S is commutative, then the graph of a becomes much more rigid.

Lemma 3.2.3. *Let S be a unital, commutative semigroup that has an element a whose graph is connected and of $\mathbf{Z}$ type. Then a can only have hairs of length at most a fixed number (in particular, there is only one possible cycle, and no infinite branches going* *into it), and if it has hairs, then a must come from the second last vertex of some hair of maximal length.*

Proof. Since the graph of a is of $\mathbf{Z}$ type, we can pick an element $a_1 \in S$ that is in the -1 grading class such that $a^k = a^{k+1}a_1$ for some k. Assume that there is a hair of length at least $k+1$ (for example, a hair on an infinite branch), then there are two elements, c and d such that $a^{k+1}c = a^{k+1}d$ and $a^k c \neq a^k d$. Then, applying a_1 to both sides we get, $a_1 a^{k+1} c = a_1 a^{k+1} d \Rightarrow a^{k+1} a_1 c = a^{k+1} a_1 d \Rightarrow a^k c = a^k d$, thus we have a contradiction. So a must come from a hair of maximal length, in particular there are no branches of infinite length, with a being a hair of the second furthest length away from the cycle. □

Theorem 3.2.1. *(Classification of Bounded, Pointed, Connected, Commutative, Semigroup Operators)*

Let (A, a) be a pointed and bounded basis map operator whose graph is connected. Then A is a semigroup operator from a commutative, unital, and bounded semigroup with multiplication coming from a if and only if A has a cycle and the conclusions of lemmas 3.2.2 and 3.2.3 are satisfied. That is, if and only if one of the following cases holds:

1. *The cycle of A is of $\mathbf{N}$ type with a as an element in the 1 grading class which has a predecessor.*

2. *The cycle of A is of finite type with a finite number of hairs going into it and a is a hair that is the second furthest away from the cycle, which has a predecessor.*

3. *The cycle of A is of $\mathbf{Z}$ type with hairs of length at most some fixed number, with a as a hair that is the second furthest away from the cycle, which has a predecessor.*

Proof. We need to show that any possible graph given in the above two lemmas is indeed a left multiplication operator for some commutative, bounded semigroup.

We begin with the case of an **N** cycle. We let a be an element in the first grading class, which has a predecessor, and form an **N** cycle from it (the set consisting of the predecessor along with it's images under powers of A). Using the powers of left multiplication by a, we can define multiplication for this cycle part (this is a singly generated semigroup, and so is commutative and associative). Next, we extend the defined multiplication to the rest of the basis elements over the course of a few steps.

We form a set M by looking at each power a^i, and picking some hair going into a^i that is maximal in length (if there are no new hairs on a^i, then we just skip it and go onto the next one). We order M via $x < y$ if the length of the hair y is longer than the length of the hair x, and then turn this into a total ordering by assigning an arbitrary, fixed total ordering on each subset M_n consisting of the elements whose length as a hair is n. Now, we extend the multiplication from the powers of a to include multiplication on the elements M and their successors under A as follows:

For each $x, y \in M$, if $x \leq y$, then $xy = yx = a^k y$, where k is the a-grading class of x. Likewise, for $x \leq y$, $m, n \in \mathbf{N}$, $(a^m x)(a^n y) = a^{m+n+k} y$. By definition, this multiplication is commutative, and it is associative since for $x \leq y \leq z$, $x(yz) = x(a^l z) = a^{k+l} z$ and $(xy)z = (a^k y)z = a^{k+l} z$. By commutativity, all the other possibilities follow as well from the above case. There is a slight technical concern in that in an intermediate step, we could get that say, $x(yz) = x(a^l z)$, but $a^l z$ is now in the cycle of A, so $a^l z = a^{l+j}$ for some $j \in \mathbf{N}$. This however is a minor concern due to the fact that the length of z as a hair is bigger than the length of x as a hair, so $x(a^l z) = a^{l+j} x = a^{l+j+k}$, while $(xy)z = (a^k y)z = a^{k+l} z = a^{k+l+j}$.

Now we wish to extend this multiplication to all the basis elements. For each $x \in M$ (which is a hair of a^i), we define multiplication of each hair y of a^i by z, a hair of a^j, as $yz = zy = a^l z$ if x is less than the representative corresponding to z in M, and $yz = zy = a^m y$ (where m is the a-grading class of z) if x is bigger. By the same logic as before, all composable products associate, but multiplication of two hairs of the same element in the cycle is not well-defined. For these products cd, we define $cd = dc = a^{s+t-k}x$, where $s, t \in \mathbf{N}$ are the a-grading classes of c and d, x is their corresponding element in M, and k the a-grading class of x. For any product $c(dz)$ now, if $x \leq z$ then $c(dz) = c(a^t z) = a^{s+t}z$, while $(cd)z = (a^{s+t-i}x)z = a^{s+t}z$. Likewise, if $x \geq z$, then $c(dz) = c(a^l d) = a^{s+t+l-i}x$, while $(cd)z = (a^{s+t-i}x)z = a^{s+t-i+l}x$. Finally, if g also has x as its corresponding element in M, and has a grading of $q \in \mathbf{N}$, then $c(dg) = c(a^{t+q-i}x) = a^{t+q-i+s}x$ while $(cd)g = (a^{s+t-i}x)g = a^{s+t-i+q}x$. This defines multiplication for the remainder of the possible products, and so we have a semigroup S with $A = L_a$.

Now suppose that the graph has a finite cycle with hairs. Then there must be a bound on the lengths of the hairs, and by boundedness of the operator, the number of hairs as well. Thus, the number of vertices themselves must be bounded, so S has to be finite. To construct the desired semigroup, we preform the construction given in the above on a modified basis map operator A' of A. This modified basis map operator is the same as A, except the largest power of a (say a^k) does not go into a smaller power of a (say a^l), but rather continues on into an $\mathbf{N}$ cycle, and then preform the construction given in the $\mathbf{N}$ cycle case on this A', to get a semigroup which we will call S'. On this semigroup, we then form a quotient semigroup by the relation $a^{l+1} = a^k$. Since a is on a hair of maximal length, anything in S' times a^k is again a power of a, larger than or equal to a^k, and so for any product of elements in S', if the relation $a^k = a^{l+1}$ can be used to make two elements in S equal that were not equal in S', then the elements would have to be powers of a larger than or equal to a^k. Likewise, by taking S' with the functor G_a, and then applying the induced quotient relation (with the same idea as Proposition 3.2.4, where we pull back the grading class to the cycle of the semigroup, and then push forward under the homomorphism and take the induced grading) gives a singly generated semigroup with a $l - k$ cycle. Since the quotient map from S' to S is a semigroup homomorphism, and G_a is a functor, we must have that applying the quotient map first and then the induced G_a must give the same result. Thus, left multiplication by a in S must have an $l-k$ cycle exactly and not something strictly smaller, and so S is indeed the semigroup we wanted to construct.

Finally, we wish to handle the **Z** cycle case. Here, we apply the same idea as in the **N** cycle case, but only to a subset of the basis, and then expand the multiplication. Specifically, we pick take the lowest power of a (call it a^k) that goes into the cycle, and then form A', where A' is A, but restricted to the hairs that go into a^k after enough iterations, and powers of a. Applying the construction of the **N** cycle case, we obtain a commutative semigroup on these elements, and now wish to extend it to the other elements. First, we extend the multiplication to include all the elements of the cycle by saying that for any element d, and z in the cycle, dz equals the element in the cycle whose a-grading is the a-grading of d plus the a-grading of z. Thus, the cycle becomes an ideal of the semigroup, as shown in Figure 3.3.

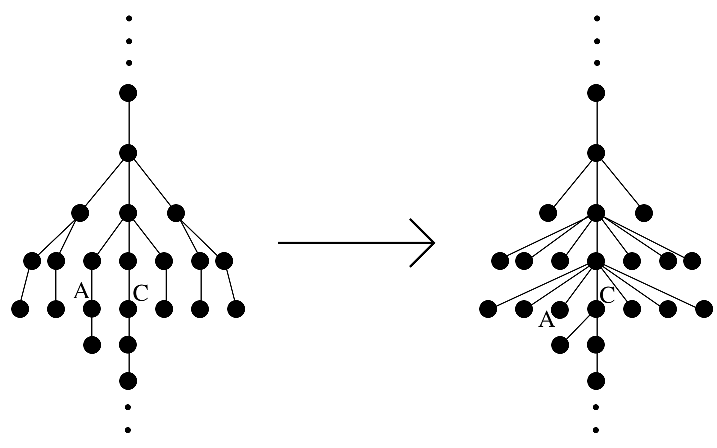

Figure 3.3: An infinite Z cycle with L_A on the left and L_C on the right

Now, we extend the multiplication to include the remaining set of basis elements (which we call N). For any two elements in N, we define their product to be the element in the cycle corresponding to the sum of their a-gradings. For an element x which is outside of N, and d in N, we define $xd = a^m d$, where m is the a-grading class of x (here we note that by construction, either the grading class of x is positive, or x is in the cycle, so this is well-defined). This gives us a commutative multiplication, so we need to only show that it is associative.

For any three product, we break it up into cases based on what elements are in N. If all three are in N, then the multiplication associates since after one multiplication we get something in N times something in the cycle, which results in something in the cycle, whose a-grading corresponds to the sum of the a-gradings of each element in the product. This is true regardless of the order of the multiplication, and so it associates. If only two of the products are in N, then either the first multiplication will be between the two elements in N, in which case we get something in the cycle times something outside of N, which gives the element in the cycle with the corresponding a-grading, or the first multiplication is between something in N and something outside of it, which will result in something from N (from the definition between multiplication of things in N and outside of N), giving us the second product to be between two things in N, with the a-grading corresponding to the sum of the a-gradings. Regardless of the order here, we still get an element in the cycle with the same grading class, and so we must get the same result. Finally, if only one of the elements is in N, then both orders of multiplication will send the product to the element in N, but multiplied by a^k, where k is the sum of the a-gradings of the elements outside of N, and so these products also associate. Thus, this multiplication is associative and so forms a commutative semigroup.

Lastly, we verify that these given semigroups are indeed bounded if A is a bounded operator. In the case of a finite cycle, we get a finite semigroup, and so the multiplication is bounded. If the operator has an $\mathbf{Z}$ cycle, then there is a bound on the number of elements in each grading class (the maximum length of hairs times the cardinality norm of A), and so even if every element of a grading class is sent to one element by some multiplication, the cardinality norm of this multiplication would still be less than or equal to the bound of the number of elements in the grading classes. Thus, each multiplication operator is not only bounded, but also uniformly bounded. Finally, in the case of an

N cycle, semigroup constructed above, though useful for the finite and **Z** cycle cases, will not always be bounded. We will show that in this case, A can still be realized as a semigroup operator in a bounded semigroup by modifying the above construction.

First, we take the operator A and extend it to a new operator A', which, for r the cardinality norm of A, has exactly r basis elements going into every strictly positive grading class basis element under applying A. For example, if the cardinality norm of A is 2, then each basis element in the positive grading classes of A' would have two elements going into it under A', and would be equal to $A^k d$ for some 0 a-grading class element d, and $k \in \mathbf{N} \cup \{0\}$, as shown in Figure 3.4.

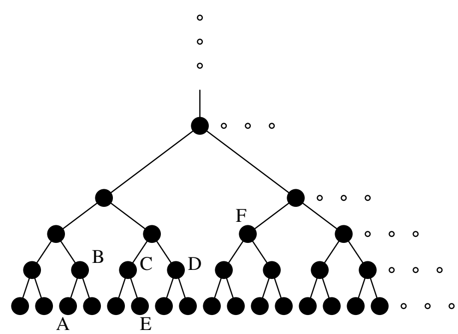

Figure 3.4: An infinite **N** cycle with B as the multiplication.

We then preform the construction given above for **N** cycles to get a semigroup S'. Since M here is the set of 0 a-grading elements, and by construction $xy = a^k y$ for $x < y$, $x, y \in M$, we have that S' restricted to the original basis elements of A is a subsemigroup, and so we need only show that S' is a bounded semigroup. First, we note that left multiplication by each 0 a-grading class element is bounded. Indeed, if c is a 0 a-grading element, then L_c acts as the identity on any branch of hairs bigger than c in the ordering. However, for m the cardinality norm of A' there are only m^k (k the length of c as a hair on the cycle) 0 a-grading elements on which L_c does not act as the identity. Thus, L_c is bounded in cardinality norm by m^k when we restrict the domain to 0 a-grading elements, and then bounded by m^{k-1} when we restrict our domain to 1 a-grading elements, and so on until we look at the k a-grading elements, on which it acts as the identity. Thus L_c is a bounded operator, and in general, for every 0 a-grading element d, L_d is bounded. To show that the entire semigroup is bounded, we just note that every other element b in the semigroup S' is equal to $a^k d$ for some $k \in \mathbf{N}$ and d a 0 a-grading element, and thus $L_b = (L_a)^k L_d$. Since both of these operators are bounded, L_b is bounded, and so S' is a bounded semigroup. Thus, if a commutative, connected, semigroup operator is bounded, then it can be realized from a bounded semigroup. □

The possibly unbounded semigroup case follows immediately, since the forward directions of the above proof did not require the boundedness property of the semigroup (with the exception of course that the basis map operator was bounded).

Corollary 3.2.1. *(Classification of Pointed, Connected, Commutative, Semigroup Operators)*

Let (A, a) be a pointed (possibly unbounded) basis map operator whose graph is connected. Then A is a semigroup operator from a commutative, and unital semigroup with multiplication coming from a if and only if one of the following cases holds:

1. *The cycle of A is of finite type with a maximal length to the hairs going into it and a is a second furthest hair away from the cycle, which has a predecessor.*

2. *The cycle of A is of $\mathbf{N}$ type with a as an element in the 1 grading class, which has a predecessor.*

3. *The cycle of A is of $\mathbf{Z}$ type with hairs of length at most a fixed length, with a as a second furthest hair away from the cycle, which has a predecessor.*

As well, the possibly non-unital follows.

Corollary 3.2.2. *A connected basis map operator A can be realized as a L_a with a in some connected, commutative semigroup S if and only if it can be realized in a unital one. For the pointed basis map operator (A, a), a has to be a hair whose length is the maximum length k, or $k - 1$.*

In the previous discussion about semigroup matrices and their corresponding graphs, there was some ambiguity in the infinite case about the cycle of a graph since for infinite cycles, a cycle of a graph that had a **Y** branch or an **N** cycle with a bunch of finite hairs coming out of it need not be unique. The above lemma gives the cycle a clear definition for connected pointed semigroup operators as the unique infinite cycle in a **Z** cycle if S is commutative, or as the identity along with the powers of the basis map operator from which the graph arises (the **N** cycle case). However, unlike in the finite and **Z** cycle cases, the cycle in the **N** cycle case need not be an ideal[12] of the connected subsemigroup generated by the element the multiplication comes from. This in particular highlights the main difficulty in constructing semigroups in this case, since not having the cycle as an ideal limits us from easily defining multiplications on the various hairs of the basis map operator.

3.2.5 Pointed, Connected, and Non-Commutative Semigroup Operators

From the previous discussion, we have a classification of the connected basis map operators that come from commutative semigroups, and in this section we will deal with the non-commutative case. While before we had some lemmas for getting a grasp on the structure of commutative semigroups, in the non-commutative case we have fewer tools to work with.

In both the finite cycle and **N** cycle type cases, all connected, semigroup operators were possible as left multiplications in a commutative semigroup. Indeed, in the finite dimensional case, the only real time commutativity became an issue was when one considered the orders of the cycles of the various components. However, in the case of a **Z** type connected semigroup operator, we could not deduce that a **Z** cycle with an infinite

[12] There we note that an ideal in a semigroup S is a subsemigroup M for which $SM \subset M$ and $MS \subset M$.

branch going into it would be impossible in the non-commutative case. In this section, we will work towards classifying these types of operators.

The main case to consider is when we have a connected graph of **Z** type such that there are no branches of hairs of finite length (that is, every element has a predecessor under the basis map operator, or equivalently, the basis map operator is surjective), and we want the semigroup to have an identity. For this case however, the requirements on the basis map operator are not as simple.

Proposition 3.2.6. *Let S be a unital and connected semigroup with the graph of L_a being connected and surjective, but not injective. Then S contains a subsemigroup isomorphic to the bicyclic semigroup $B =< a, h| \; ah = e >$.*

Proof. First, we note that since L_a comes from a connected semigroup, the connected component of a in the graph of L_a has a cycle, and since L_a is surjective, the cycle can not be a **N** cycle. Likewise, the cycle can't be finite since surjectiveness implies that any hair of positive length would have to have an infinite branch of hairs going into it, making it impossible to be a semigroup operator, while if it has no hairs, it would have to be injective. Thus, the cycle of L_a must be a **Z** cycle, and it would have to have at least one infinite branch of hairs going into it at some point (since it is not injective). Moreover, a must be such an element at some intersection due to Lemma 3.2.2. Thus, a must be at a vertex on the graph with e as one predecessor, h as a predecessor of e, and ha as another predecessor of a (since $a(ha) = (ah)a = ea = a$).

The subsemigroup generated by a and h in S must be the bicyclic semigroup $B =< a, h| \; ah = e >$, or a quotient semigroup of it. We now show it must be the former. Suppose there is some relation in the subsemigroup generated by a and h, say, $h^m a^k = h^{m-k+l} a^l$ for some $m, k, l \in \mathbf{N} \cup \{0\}$ (note that the relation must be between elements in the same grading class under a, otherwise the resulting subsemigroup would not be of **Z** type, and here h is in the -1 grading class, so the class of these two elements is $k - m$). Now, k must be either strictly bigger or strictly smaller than l, otherwise the relation is trivial. Without loss of generality, we will assume that k is smaller than l.

Since $k < l$, we right multiply the relation by h^k, which gives,

$$h^m(a^k h^k) = h^{m-k+l}(a^l h^k) \Rightarrow h^m = h^{m-k+l}a^{l-k}$$

$$\Rightarrow a^m(h^m) = a^m(h^{m-k+l}a^{l-k}) \Rightarrow e = h^{l-k}a^{l-k}$$

$$\Rightarrow a^{l-k-1}eh^{l-k-1} = a^{l-k-1}(h^{l-k}a^{l-k})h^{l-k-1} \Rightarrow e = ha.$$

This contradicts the fact that ha was different from e, and so the subsemigroup generated by a and h can only have the relation $ah = e$. Thus, S must contain a subsemigroup isomorphic to the bicyclic semigroup. □

We remark that the above theorem states that the bicyclic semigroup is the “simplest” example of a connected semigroup of $\mathbf{Z}$ type with no hairs. While there are more examples of such connected semigroups (and more possible types of basis map operators that arise as semigroup operators), we will have to leave this question for a later time, due to their complicated nature.

3.3 General Case

3.3.1 Finite

While the case of connected basis map operators in non-commutative semigroups is rather complicated, the more general cases for basis map operators in commutative or finite semigroups is much more tangible. We start with the case of finite semigroups.

Theorem 3.3.1. *Let A be a basis map operator over a finite dimensional Hilbert space, then A can be realized as a L_a for a in some unital semigroup S if and only if the orders of the cycles in the cycle decomposition have a common multiple and the length of hairs on all the cycles is less than or equal to the maximum hair length on some cycle of maximum order.*

Proof. First we show the forward direction. Let $A = L_a$ for some a in a semigroup S. By finiteness of the Hilbert space, we know that each component of the graph of L_a has a cycle, so L_a has a cycle decomposition. Now, for k the length of a as a hair on its cycle, a^{k+1} is in a cycle of order n, $a^{k+1+n} = a^{k+1}$, so for any $c \in S$, $a^{k+1+n}c = a^{k+1}c$. Thus, $a^{k+1}c$ is in the cycle of the component of c in the graph of L_a, meaning that hair lengths on all the cycles is bounded by the length of the hairs on the component of a. Moreover, since $a^{k+1+n}c = a^{k+1}c$, the order of the cycle of the component of c must divide n. This proves the forward direction.

For the reverse direction we wish to construct a semigroup S with $L_a = A$ for some $a \in S$. We take a hair going into the largest cycle that is the furthest away from the cycle, call it e and let a be its image under A. Next, we fill in the rows of the multiplication table with the powers of a as the powers of A, like done before in the injective basis map operator case, and call this the a-semigroupoid. Associativity of this semigroupoid is equivalent to the condition that $L_{ab} = L_a L_b$ for the composeable products ab, meaning that $L_{aa^l} = L_a L_{a^l} = L_a^{l+1}$, $l \in \mathbf{N}$, as mentioned in Lemma 3.1.4. Letting k be the length of the hair e and n the order of the cycle that a goes into, for $l < k+n$ we have associativity by definition, however, for $l \geq k+n$ we have that $a^{k+n} = a^k$, so we must show that $L_{a^k} = L_{a^{k+n}} = L_a L_{a^{k+n-1}}$. Since we have that all the hairs of A are of length k or less, we have that after taking L_a^k, we have that all the hairs have gone into their respective cycles, and so, like in the case of finite injective basis map operators, by the fact that the orders of the other cycles divides n, we have that applying A n more times gives us $L_{a^{k+n}} = L_{a^k} = L_a L_{a^{k+n-1}}$.

Now we extend the a-semigroupoid to the other elements in its connected component. To do this, for f, x in this component, we define fx the same way as it was defined in Theorem 3.2.1. As shown before, this is associative, so we extend these left multiplications to include the rest of the components. Here, we define fd, where f is in the connected component of a and d in any other component as $a^k d$, where k is the k in the definition of $fx = a^k x$ used in Theorem 3.2.1, for multiplication betweens hairs x that are larger than f on the connected component of a. This multiplication is associative on this semigroupoid, since there can be only one x in any composable product, where x is in another component. So we can only get a product of the form $f(dx) = f(a^k x) = a^{k+l}x$, where l is the corresponding k value of f in the definition of its multiplication on the connected component, and likewise, $(fd)x = (a^k f)x = a^{k+l}x$ (assuming that $f > d$ on the hair ordering used in the construction, but similarly follows for $f < d$).

Thus the a-semigroupoid has been extended to the connected component of a, so we need only extend the it to the rest of the elements. Like in the injective case, we define $ij = i$ for i outside the connected component of a. To show that this is associative, we note that any extended i, $L_i L_x = L_i = L_{ix}$, and also $L_a L_i$ will equal the matrix with the i column of L_a in all the columns, which equals L_{ai}. Hence, this multiplication is defined everywhere, associative, and has that $A = L_a$, so the reverse direction is done as well. □

Corollary 3.3.1. *Let A be a basis map operator over a finite dimensional Hilbert space, then A can be realized as a L_a for a in some (not necessarily unital) semigroup S if and only if the orders of the cycles in the cycle decomposition have a common multiple and the length of hairs on the smaller cycles is less than or equal to length of some hair of maximum length plus one on some largest cycle.*

Proof. Adjoining on an identity to the semigroup gives us a unital semigroup and by Theorem 3.3.1, the lengths of the hairs on L_a in the unitization is less than or equal to the length of the maximum hair length on the connected component of a in the graph of L_a. Removing the identity, we get that the hairs on the other cycles must be at most one more than the maximal hair length of the a component of the original L_a, giving the forward direction.

For the reverse direction, we note that the semigroup constructed in Theorem 3.3.1 has the property that the only things that multiply to get the identity is the identity times the identity (with the exception of L_a being cancellative) since the multiplication was defined to send everything into one of the cycles. Thus, we can take the L_a and add a new element that is sent into a, perform the construction, and then remove the identity again, giving us the desired semigroup S. In the case that L_a is cancellative, then there are no hairs on the cycles, and the unital construction goes through immediately. □

Similarly, the analogous theorem for commutative semigroup is also true.

Theorem 3.3.2. *Let A be a basis map operator over a finite dimensional Hilbert space, then A can be realized as a L_a for a in some unital commutative semigroup S if and only if the orders of the cycles in the cycle decomposition have a common multiple and a common divisor and the length of hairs on all the cycles is less than or equal to length of the hair of maximum length on some largest cycle.*

Proof. From Theorem 3.3.1, we need only show that the added condition of a common divisor cycle is both necessary and sufficient.

For the forward direction we apply the same idea as before for the injective case. Suppose we have a semigroup that is commutative, and let c and d be from cycles of orders m and n respectively. Then for some power k of a, both $a^k c$ and $a^k d$ will be in their respective cycles. Then, $a^m(a^k c)d = (a^k(a^m c))d = a^k cd$, but $a^n a^k cd = (a^n a^k d)c = a^k dc = a^k cd$. Thus, the element $a^k cd$ must be in a cycle whose order is divisible by both m and n.

Next, we have to show that having a divisor cycle is sufficient, granted that the length of all the hairs are less than or equal to the length of a hair on one of the largest cycles. We perform the same construction of the a-groupoid as done in Theorem 3.3.1, and note that we just need to extend it to the rest of the multiplication table so that it is commutative.

Like in the cancellative finite dimensional basis map operator case, we pick a distinguished element from each of the cycles in the other components, and let the divisor cycle's distinguished element be called m. We define ij for these new elements to be am, while $ia = ai$. Now we extend the grading equivalence classes to the other components in the obvious way: x is equivalent to y if and only if $a^l x = a^l y$ for some $l \in \mathbf{N}$. Every element now will be equivalent to some element in a cycle and we define multiplication on all the other elements to act like their corresponding distinguished cycle elements via the expanded grading equivalence classes.

For associativity, if we take a three product consisting of elements from the cycles, then it will associate since the multiplication for this construction restricted to the cycles is the same as the finite and cancellative case for commutative semigroups. Thus, any three product of elements from the connected components that doesn't contain a will also associate since these left multiplications act like their respective elements in their cycles. Thus, we need only show that a product containing one or two elements from the a component and the rest from the others will associate (we know that three elements from the a component will associate since our construction here, when restricted to this component is the same as our construction for a connected, commutative semigroup).

First, let x be in the a component, and y, z in some other component. Then, $x(yz) = x(a^l c a^k d)$ where y acts like $a^l c$, z acts like $a^k d$, and c, d the distinguished elements on their respective cycles. However, $x(a^l c a^k d) = x(a^{l+k+1} m)$, where m is the distinguished element of the divisor cycle from the construction. Letting n be the grading equivalence class of x, we have that this equals $a^{n+l+k+1} m$. For the other product, the equivalence class of (xy) will be in the same one as $a^{n+l} c$, so when we multiply it by z, it will become $a^{n+l+k} cd = a^{n+l+k+1} m$, so these products associate. Likewise, because of commutativity of the multiplication, this case extends to the case when one of x, y, z is from the a component, and the others not.

For the case that two elements x and y are both in the a component, and z is not, then $x(yz)$ will associate since it's a composable product in the a-semigroupoid. Finally, using commutativity again, this covers the case when exactly two of x, y, z are in the

a component and the other not, and so the product $x(yz)$ must associate. Thus, the existence of a divisor cycle and the maximality of the hairs on a largest cycle are both necessary and sufficient for finite, commutative, and unital semigroups. □

In the exact same way as the corollary after the finite unital semigroup case, we also have the finite, non-unital, commutative semigroup case as a corollary of the unital one.

Corollary 3.3.2. *Let A be a basis map operator over a finite dimensional Hilbert space, then A can be realized as a L_a for a in some (not necessarily unital) commutative semigroup S if and only if the orders of the cycles in the cycle decomposition have a common multiple and a common divisor and the length of the hairs on all the cycles is less than or equal to length of the hair of maximum length plus one on some largest cycle.*

3.3.2 Infinite and Possibly Non-Bounded, With Full Cycle Decomposition

The easiest case to deal with in infinite dimensions is when the semigroup can be unbounded. As we have seen from the injective case, having a bounded condition on the semigroup can be rather limiting in terms of what kind of operators we can get, so removing this condition will make constructing a desired semigroup much easier. As well, we will consider in this section only what happens in the case that each component of the basis map operator has a cycle.

The first case considered is for non-commutative, unital semigroups, and the basis map operator has only finite cycles.

Theorem 3.3.3. *Let A be a basis map operator over an infinite dimensional Hilbert space which has a full cycle decomposition. If A has only finite cycles, then it can be realized as a L_a for a in some unital (possibly unbounded) semigroup S if and only if the orders of the cycles in the cycle decomposition have a common multiple and the length of* *hairs on the smaller cycles is less than or equal to length of the hair of maximum length on some largest cycle.*

Proof. We apply the same idea as the finite dimensional case. For the forward direction, we note that a has to be connected to some cycle, and so a^n is in a cycle for some integer n. Thus, $a^{n+k} = a^n$ for k the length of this cycle. Thus, for any element in the cycle decomposition of A, if we multiply it by A^{n+k} we must get the same as multiplying it by A^n, and so the order of all the cycles must divide k. Next if we assume that some other cycle has a hair of greater length than the ones on any largest cycle (call this element x), then there is a non negative integer l such that $a^l a$ is in a cycle, while $a^l x$ is not. Thus, for some k, $a^{l+k}a = a^l a \Rightarrow a^l x = a^{l+k}x$. But $a^{l+k}x$ can not be $a^l x$ since if it did, then for some i, $a^{l+ik}x$ would be in the cycle of the component of x, but $a^{l+ik}x = a^{l+(i-1)k}x = ... = a^l x$, contradicting the fact that $a^l x$ is not in the cycle. Finally, there can't be an infinite branch going into the cycle of a since if there was, then there would be integers l, k such that $a^l = a^{l+k}$, while using the above idea we can pick an element y that is more than $l + k$ elements from the cycle, giving us that $a^l y \neq a^{l+k}y$.

For the reverse direction, we apply the same construction as in the finite dimensional connected case, and extend it to all the cycles here. Any three product here now will reduce down to the subsemigroup generated by the three elements, which will be finite by the construction, so by Theorem 3.3.1 it will be associative. Thus, this constructed semigroup is indeed associative (though incredibly unbounded). □

As expected, if one wishes for the semigroup to be commutative, then the additional requirement is the existence of a cycle whose order divides the order of all the other cycles.

Theorem 3.3.4. *Let A be a basis map operator over an infinite dimensional Hilbert space, with A having a full cycle decomposition, and only finite cycles. A can be realized as a L_a for a in some unital (possibly unbounded) commutative semigroup S if and only if* *the orders of the cycles in the cycle decomposition have a common multiple and a common divisor, and the length of hairs on all the cycles is less than or equal to length of a hair of maximum length on some largest cycle.*

Proof. First we show the forward direction. Let L_a be a semigroup operator from an element a in a commutative semigroup S. Like in previous arguments, multiplying an element in an m cycle by an element in an n cycle results in an element that is in a cycle of an order that divides both m and n. Thus, there must be a cycle whose order divides the order of all the other cycles, and also a cycle whose order is a common multiple of all the others as well.

For the reverse direction, we just apply the same construction as in Theorem 3.3.2, but use it for infinitely many cycles. For any three basis elements from A, there will be a finite stage of the construction that will contain all three, and so the product will associate in this bigger semigroup. Thus, this A can be realized as a L_a for some a in a commutative, unital semigroup. □

Again, like in the finite dimensional case, the corresponding theorem without the unital condition on the semigroup follows as a corollary.

Corollary 3.3.3. *Let A be a basis map operator over an infinite dimensional Hilbert space with A having a full cycle decomposition, and only finite cycles. Then A can be realized as a L_a for a in some (possibly unbounded and non-unital) semigroup S if and only if there is a cycle whose order is a common multiple of the orders of all the cycles, and the length of the hairs on all the cycles is less than or equal to the length of a hair of maximum length plus one on some largest cycle. Likewise for a commutative semigroup, also requiring a cycle whose order is a common divisor is both necessary and sufficient.*

Proof. As before, the above constructions for the commutative and non-commutative semigroups do not have any non-trivial product decompositions for the identity. Thus, we can just add an appropriate hair to the basis map operator, construct the desired semigroup, and then remove the identity. □

Next, we consider the case when some components may be of infinite type, but all have a cycle. Since connected non-commutative semigroups of $\mathbf{Z}$ type have a rather complicated structure, we will only consider the commutative case.

Theorem 3.3.5. *Let A be a basis map operator over an infinite dimensional Hilbert space, with A having a full cycle decomposition. Then A can be realized as an L_a for some a in a commutative semigroup S if and only if the orders of the cycles have a common multiple and a common divisor (with the understanding that an $\mathbf{N}$ cycle is "a multiple of" a $\mathbf{Z}$ cycle, and both are multiples of any finite cycle), and if the multiplication does not have an $\mathbf{N}$ cycle, then the length of all hairs is bounded by some finite number, which is the maximum hair length on some cycle of largest order.*

Proof. We show the forward direction first and assume that $A = L_a$ for some a in a commutative semigroup S. By Theorem 3.3.4 we may assume that there is at least one infinite cycle for the basis map operator, and as such, it has a cycle whose order is maximal. We also note that if there are no finite cycles, then there is a cycle whose order divides all the orders of the cycles (either there is a $\mathbf{Z}$ cycle whose order divides any infinite cycle, or there are only $\mathbf{N}$ cycles, in which all cycles divide themselves). If there are finite cycles, then the product of any elements with an element from another finite cycle must be in a finite cycle whose order divides it, and so there must be a cycle whose order divides the orders of all the other cycles. For the case of there not being an $\mathbf{N}$ cycle, by Theorem 3.3.4 we can assume that there is a $\mathbf{Z}$ cycle, and by Corollary 3.2.1 we know that one of these cycles (the one in the component that the multiplication comes from) must have a hair of maximal length (call this length k) away from the cycle, and this element is the identity of the semigroup. Now, take an element f that is in the -1 grading class of a, then $a^k = a^{k+1}f$. If there is a hair of length greater than k on any of the finite cycles, then in particular there is one that is of length $k+1$ (call this element h), so $a^k h$ is not in any cycle, but $a^{k+1}h$ is. Now, $a^k h = a^{k+1} f h = f(a^{k+1}h)$, however, $a^{k+1}h$ is in a cycle, so for some $l \in \mathbf{N}$, $a^{k+1}h = a^{k+1+l}h$. Thus, $f(a^{k+1}h) = f(a^{k+1+l}h) = a^{k+1} f a^l h = a^{k+l}h$. So $a^k h$, which is not in any cycle equals $a^{k+l}h$, which is in a cycle, giving us a contradiction. Finally, consider the case that there is a hair of length $k+1$ on another $\mathbf{Z}$ cycle component. Then there are two elements h, z in this component such that $a^{k+1}h = a^{k+1}z$, but $a^k h \neq a^k z$ (h is the hair of length $k+1$, and z is in the cycle of the component), but since $a^k = a^{k+1}f$, $a^{k+1}h = a^{k+1}z \Rightarrow f a^{k+1} h = f a^{k+1} z \Rightarrow a^k h = a^k z$, another contradiction. Thus the forward direction is done.

Now we show the reverse direction and construct a commutative semigroup S from which A arises. Letting a be the image of a hair of maximal length in a maximal cycle, we preform the same construction as in Theorem 3.2.1 for the connected component of a (which we denote as S_0) but extend it to include the other cycles as well. For $x \in S_0$, $y \in S - S_0$, define xy and yx as $a^{G_a(x)}y$. In the case that the connected component of a is an $\mathbf{N}$ cycle or a finite cycle the above definition works fine, but if the connected component is a $\mathbf{Z}$ cycle, and x has a negative grading, then it is not well defined. For this case, we instead define xy as the element y' in the cycle of the connected component of y such that for some sufficiently large $k \in \mathbf{N}$, $a^k y' = a^{k+G_a(x)}y$.

Next, we use the same idea as in Proposition 3.1.3 to define the multiplication on the rest of the elements by selecting a distinguished element y_i from all the other cycles in the cycle decomposition, and then extending the grading functor to these components. Explicitly, this extension is given by $G_a(y) = j$ if and only if $a^k y = a^{k+j} y_i$, where y_i is the distinguished cycle element in the connected component of y. Finally, the for $x, y \in S - S_0$, we define xy as $a^{G_a(x)+G_a(y)}c$, where c is the distinguished element in the common divisor cycle.

This multiplication is now totally defined, so we need only show that it is associative. From Theorem 3.2.1 we know that for $x, y, z \in S$, $x(yz) = (xy)z$. If $x, y \in S$ with both a-gradings non-negative, and $z \in S - S_0$, then $x(yz) = x(a^{G_a(y)}z) = a^{G_a(x)+G_a(y)}z$, while $(xy)z = kz = a^{G_a(x)+G_a(y)}z$, where k is some element in S with a a-grading of $G_a(x) + G_a(y)$.

If $x, y \in S$ with at least one of their a-gradings negative, and $z \in S - S_0$, then $x(yz) = x(a^{G_a(y)}z_0) = a^{G_a(x)+G_a(y)}z'$ (where z' is in the cycle of the connected component of z and z_0 is either z or z' depending on if the a-grading of y is negative), while $(xy)z = kz = a^{G_a(x)+G_a(y)}z'$, where k is an element in the cycle of the connected component of a with a a-grading of $G_a(x) + G_a(y)$ (here we note that the construction used in Theorem 3.2.1 makes the multiplication of a negative grading element with any other element in the connected semigroup part of the cycle automatically, and so kz will indeed be in the cycle even if $G_a(x) + G_a(y) > 0$).

Next, if $x \in S$, and $y, z \in S - S_0$, then $x(yz) = x(a^{G_a(y)+G_a(z)}c) = a^{G_a(x)+G_a(y)+G_a(z)}c$, while $(xy)z = (a^{G_a(x)}y_0)z = a^{G_a(x)+G_a(y)+G_a(z)}c$, where y_0 is either y (if the a-grading class of x is non-negative), or the grading equivalent element in the cycle of y (if the a-grading class of x is negative). Finally, if $x, y, z \in S - S_0$, $x(yz) = x(a^{G_a(y)+G_a(z)}c) = a^{G_a(x)+G_a(y)+G_a(z)}c$, while $(xy)z = (a^{G_a(x)+G_a(y)}c)z = a^{G_a(x)+G_a(y)+G_a(z)}c$. The rest of the possibilities for products follow from commutativity, and so the defined multiplication is associative, and hence S is a semigroup with A arising as L_a. □

3.3.3 Infinite and Possibly Non-Bounded, Without Full Cycle Decomposition

Now, the case of a basis map operator without all connected components having a cycle should be examined. These operators require more investigation to understand how they behave inside a semigroup, and so we start with some lemmas.

Lemma 3.3.1. *Let S be a semigroup, and $a \in S$. If the graph of L_a has a component that does not have a cycle, then L_a must have a component that is an* **N** *cycle. Moreover,* *if $c \in S$ is in a component that does not have a cycle, then any product cz can not be in an* $\mathbf{N}$ *cycle, and in particular, if $b \in S$ is in a component of L_a with a* $\mathbf{Z}$ *cycle (or a finite cycle of order n) then the product bc must be in a component with a cycle (or a finite cycle whose order divides n if b was from a finite cycle of order n).*

Proof. For the first assertion, we have the possibilities that a is on a component with an $\mathbf{N}$ cycle, it is on a component with a $\mathbf{Z}$ cycle, it is on a component with a finite cycle, or it is on a component without a cycle. If the component of a had a finite cycle, then for some $k, l \in \mathbf{N}$, $a^k = a^{k+l}$, and so $(L_a)^k = (L_a)^{k+l}$. Thus, for any element in the component without a cycle, after multiplying by a on the left k times, the resulting element will have infinitely many predecessors, contradicting Proposition 3.2.2. Likewise, Proposition 3.2.5 tells us that the connected component of a must have a cycle, so we just need to show that the connected component of a can not have a $\mathbf{Z}$ cycle.

Suppose it did, then for some power k of a, a^k has an infinite number of predecessors, and in particular has a predecessor in the -1 grading class. Call this element f and take an element d on the component without a cycle that does not have any predecessors. Then $a^k d = a^{k+1}(fd)$, and likewise, for any $l \in \mathbf{N}$, $a^k d = a^{k+l}(f^l d)$. So $a^k d$ must have an l-th predecessor for any positive integer l, contradicting the fact that Proposition 3.2.2 says that all the element must have only a finite number of predecessors. Thus, a must come from an $\mathbf{N}$ cycle.

For the second part, we can do a similar trick to the above. Assume that the product cz is in an $\mathbf{N}$ cycle, where z is in some cycle. Now, cz must have only a finite length (say l) to its predecessors under L_a, so we pick a c' in the component of c such that $a^k c = a^{k+l+1}c'$. Then $a^k(cz) = a^{k+l+1}(c'z)$, so $a^k(cz)$ must have a $k+l+1$-th predecessor, which is a contradiction. Thus cz can not be in an $\mathbf{N}$ cycle.

Finally, if b is in a component with a cycle that is of $\mathbf{Z}$ type, then for some $k \in \mathbf{N}$, $a^k b$ has an infinite branch of predecessors. Hence, for any $l \in \mathbf{N}$, there is an $b' \in S$ such that $a^k(bc) = a^{k+l}(b'c)$, so $a^k(bc)$ must have a $k+l$-th predecessor, in particular, it can't have only a finite number, thus the component it belongs to must have a cycle, and moreover, one that is not an **N** cycle. In the case that b is in a component with a finite cycle, then for $k, l \in \mathbf{N}$, $k < l$, $a^k(bc) = a^l(bc)$, so bc must be in a component with a finite cycle under L_a. □

We note that in the above lemma, zc can be in an **N** cycle if z is. One reason for this is because was can still do the idempotent trick for the non-commutative case, which leads us into the next theorem.

Theorem 3.3.6. *Let A be a basis map operator with at least one of its components not having a cycle. Then A can be realized as a L_b for some unbounded (unital) semigroup S if and only if A has an* **N** *cycle.*

Proof. Lemma 3.3.1 gives the forward direction, so we need only prove the reverse implication. We pick the second element in some cycle of the **N** cycle component, and call it a. Using A, we can define L_a and all of its powers by $L_e = I$ (where e the predecessor of a), $L_a = A$, $L_{a^2} = A^2$, and so on. This gives a semigroupoid such that the left multiplications on the cycle are defined.

Next, we extend the multiplication to be defined on the rest of the connected component of a in the same way as was done in previous, noncommutative cases. Specifically, we define the multiplication as in Theorem 3.3.5, but only for left multiplication of the connected connected component of a (note that is did not need cycle elements in the other components when a was in an **N** cycle).

As before, for all elements i outside of the connected component of a, we define $ix = i$, for any x. By the same argument as in Theorem 3.3.1, this multiplication is associative and defined on the entire semigroup, so we are done. Finally, we note that if we did not want a unital semigroup, we could have just picked a to be the first element in the cycle, and the same construction without using e would give a non-unital semigroup with $L_a = A$. □

In a similar way, some parts of the commutative case can also be handled like in the finite cases.

Theorem 3.3.7. *Let A be a basis map operator with at least one of its components not having a cycle. If A has a component with a $\mathbf{Z}$ cycle or a finite cycle, then A can be realized as a L_a in a commutative (unital) semigroup if and only if, A has an $\mathbf{N}$ cycle and a component with a cycle that divides the types of all other components with a cycle.*

Proof. For the forward direction, we know that Lemma 3.3.1 requires that there is an $\mathbf{N}$ cycle, and from previous proofs we know that if c and d are in finite m and n cycles respectively, then the product cd will have to be in a cycle whose order divides both m and n. Thus, there must be a component with a cycle whose order divides the orders of all other components with a cycle, and so the forward direction is shown.

Now, we wish to prove that a basis map operator with an $\mathbf{N}$ cycle, and a divisor cycle, must exist as a left multiplication in some commutative (and unital) semigroup. We start with the same idea as before in Theorem 3.3.6, and define multiplication by a (where a is an element with a predecessor in the 1-grading class of the $\mathbf{N}$ cycle), as A. Likewise, we can continue and define multiplication on the other elements in the connected component of a with the same construction as the previous theorem. We obtain a semigroupoid that is associative for the composable products, and now wish to extend it to multiplication between elements of the other components. For this, we do the same idea as before with distinguished elements in each of the components and define multiplication between the other components by summing their corresponding, extended a-gradings, and taking the element with that a-grading in the divisor cycle as the result. For the same reasons as before, this multiplication will associate, and so gives the desired commutative semigroup. □

We are now left with the case that the basis map operator has at least one component without a cycle, and every component with a cycle is of $\mathbf{N}$ type. We establish some more lemmas regarding the possibilities in this case.

Lemma 3.3.2. *Let S be a commutative semigroup, and $a \in S$. For any element f in a component of the graph of L_a that does not have a cycle, if f^m and f^n are in the same component, then this component must have a cycle. In particular, if L_a has only a finite number of components without a cycle, then there must be a component that does have a cycle and is of type $\mathbf{Z}$ or finite type.*

Proof. We first show that if f^2 can not be in the same component as f. Suppose it was, then for some $i, j \in \mathbf{N}$, we have that $a^i f^2 = a^j f$. We note that if $i > j$, then $a^i f^2 = a^{i-j}(a^j f)f = a^{i-j}(a^i f^2)f = a^{2(i-j)}(a^j f)f^2 = a^{3(i-j)}(a^j f)f^3$, and so on. This means that $a^i f^2$ must have an infinite chain of predecessors (since $i - j > 0$), and so the component of f must have a cycle. Thus, we only have to deal with the case that $i \leq j$. The idea we wish to use now is that the elements of this component must respect a particular grading structure with respect to a, but unlike in the a component, there is no obvious reference point as to which equivalence class is which. So we will just keep checking each grading class preceding the class of f until we find the right now.

Specifically, we start by choosing some element c such that $a^k f = a^{k+1}c$ for some $k \in \mathbf{N}$. Moreover, we may assume that k is large enough that $k > i$. Then, $a^{2k+2}c^2 = a^{2k}f^2 = a^{2k-i}(a^i f^2) = a^{2k-i}(a^j f) = a^{k-i}a^j a^k f = a^{k-i+j}a^{k+1}c = a^{2k-i+j+1}c$. Since $(2k - i + j + 1) - (2k + 2) = (-i + j) - 1$, we see that $a^m c^2 = a^n c$, but $n - m = (j - i) - 1$. Thus, if $i < j$ for f, we can just replace f with the described c so that the new difference $(j - i)$ in the equation $a^i f^2 = a^j f$ is decreased by 1, and we can do so inductively until we get a difference of -1, and then use the first case on this new element. Thus, if f^2 is in the same component as f, then the component of f must have a cycle.

For the more general case that $a^i f^m = a^j f^n$, we use the same idea as the above. Without loss of generality, assume that $n < m$. If $i > j$, then $a^i f^m = a^{i-j}(a^j f^n)f^{m-n} = a^{i-j}(a^i f^m)f^{m-n}$ and so on, giving an infinite branch of predecessors for $a^i f^m$. For the case that $i \leq j$, we use a similar idea as before and take c in a previous grading class of f.

There is a k such that $a^kf = a^{k+1}c$, where $k \in \mathbf{N}$ and $k > i$, and so $a^{mk+m}c^m = a^{mk}f^m = a^{mk-i}(a^if^m) = a^{mk-i}(a^jf^n) = a^{mk-i+j}f^n = a^{mk-i+j-nk}(a^{nk}f^n) = a^{mk-i+j-nk}(a^{nk+n}c^n) = a^{mk-i+j+n}c^n$. Taking $(mk - i + j + n) - (mk + m) = -i + j + n - m$, we note that this is less than $j - i$ since $n < m$, so applying this process inductively, we can find an element d such that $a^id^m = a^jd^n$ with $i > j$, putting us back in the first case. Thus, if the component of a power of f is the same as the component of a different power of f, then the component must have a cycle.

Finally, if L_a has only a finite number of components without a cycle, then there must be a cycle of $\mathbf{Z}$ or finite type. If not, then multiplication by an element in a component without a cycle will eventually have to have a power in the same component as another different power, which contradictions what was shown above. □

The above lemma tells us in particular that the case of having only finitely many components without a cycle is only possible if there is a dividing cycle to go into, which was covered in Theorem 3.3.7. Thus we can restrict our attention to the case that there is at least one $\mathbf{N}$ cycle, and infinitely many components without a cycle.

In addition, the above also tells us that in this case in the quotient semigroup of a commutative semigroup S by the connected subsemigroup generated by a a such that L_a has a component without a cycle, the multiplication by any image of an element in a component without a cycle inside the quotient semigroup, must either have at least one $\mathbf{N}$ cycle, or at least one $\mathbf{Z}$ cycle. The next lemma shows that the former is always the case.

Lemma 3.3.3. *Let S be a commutative semigroup, and $a \in S$. For any element f in a component of the graph of L_a that does not have a cycle, there is no element $d \in S$ such that f^kd is in the same component as f^k, and the components of f^k and d both do not have a cycle.*

Proof. We proceed with a similar idea as before. Suppose that fd is in the same component as f, then for some $i, j \in \mathbf{N}$, $a^i f^k d = a^j f^k$. If $i > j$, then $a^i f^k d = a^{i-j}(a^j f^k)d = a^{i-j}(a^i f^k d)d = a^{2(i-j)}(a^j f^k)d^2$ and so on. Thus $a^i f^k d$ has finitely many predecessors, and so the component of $f^k d$ (and hence f^k) must have infinitely many predecessors. Now, if $i \leq j$, we just pick an element g in the component of d such that $a^l d = a^{l+m} g$. Then $a^l a^i f^k d = a^{l+m} a^i f^k g$, and if m is chosen sufficiently large enough, then we are in the case of $i > j$ and by the same logic, for some $n \in \mathbf{N}$, $a^n f^k d$ would have to have an infinite number of predecessors, and thus the connected component of $f^k d$ would have to have a cycle. □

In particular, this shows that in the main case we are concerned with (L_a has at least one $\mathbf{N}$ cycle, infinitely many components without a cycle, and no other components), multiplication in the quotient semigroup by the connected component of a must have only multiplications that are of $\mathbf{N}$ type when one restricts their attention to only the components without a cycle.

With further investigation, one can see that a construction in the final case of an $\mathbf{N}$ cycle and the rest of the components without a cycle part, would involve a clever definition of multiplication so that the component multiplication would respect the length of predecessors on each of the components. While this is possible, we will leave this problem for a later time.

3.3.4 Bounded and Commutative, with Full Cycle Decomposition

The final case that we will consider here is the bounded and commutative semigroup case, with every component of the basis map operator having a cycle. Non-commutative semigroups are more difficult to work with because of their lack of structure regarding the connected components, and as such we will avoid dealing with this case. As stated, in the commutative case we have a strong relationship between the connected components of an element and the multiplication on the semigroup.

Lemma 3.3.4. *Suppose S is a bounded, unital, commutative semigroup S with $a \in S$. Given $b, c \in S$, both in finite components of L_a, then the order of the cycle of bc must divide both the order part of b and c. In the infinite case, if one of b and c are in an $\mathbf{Z}$ cycle, then the product must be in a $\mathbf{Z}$ cycle.*

Proof. By lemma 3.2.3, we may replace both b and c with x and y that are in the cycles of their respective components. Then there is an $n \in \mathbf{N}$ such that $a^n(xy) = (b^n x)y = xy$ for n the order of the cycle x is in. Likewise there is an $m \in \mathbf{N}$ such that $a^m(xy) = x(a^m y) = xy$, and thus, xy must be in a component that is of an order that divides both the orders of the cycles of x and y, and therefore, b and c.

In the infinite case, let c be in the $\mathbf{Z}$ cycle. Then c has an n-th predecessor d so $a^n bd = ba^n d = bc$. Thus bc has an n-th predecessor for any $n \in \mathbf{N}$, and so bc must be in a $\mathbf{Z}$ cycle. All cases are covered by this since L_a can not have both infinite and finite cycle components if the semigroup is bounded. □

First, we classify the case of basis map operators containing only finite cycles in their cycle decompositions.

Theorem 3.3.8. *Let A be a bounded basis map operator over an infinite dimensional, separable Hilbert space, with A having only finite cycles in its cycle decomposition. Then A can be realized as an L_a for a in some bounded, unital, and commutative semigroup S if and only if there is a cycle of maximal order that is a multiple of the orders of all the other cycles, there is a cycle that divides the orders of all the cycles (with this cycle occurring infinitely many times), and the lengths of the hairs on all the cycles is less than or equal to the length of the hairs on some cycle of maximal order.*

Proof. The only difference between this theorem and the previous Theorem 3.3.4 is the necessary and sufficient condition of there being infinitely many cycles of the smallest order to give boundedness of the semigroup. First, if A arises in a bounded semigroup S, we note that the set consisting of the union of all the cycles gives us an ideal of the semigroup, and in particular is a subsemigroup of S. This subsemigroup is bounded, with the cycle part of A arising as the cycle part of L_c for some c in one of the cycles (in particular, one example would be $L_{a^{k+l}}$, where k is the maximum length of the hairs on the cycles, and $l \in \mathbf{N}$ such that the a-grading class of a^{k+l} is 1). By Proposition 3.1.7, we know that this semigroup must have its smallest cycle occurring an infinite number of times.

In a similar way, assuming that the smallest cycle occurs an infinite number of times, we can construct a semigroup for which A arises as some L_a. This can be done in a similar way as in Proposition 3.1.7, except we use a modified definition of the semigroup T used.

Here, if there is only a single cycle type of maximal order, then we take this component (which we will denote M) and we add a single copy of each of the other cycle types of A to form a basis map operator A'. If there are two or more cycles of the maximal order, then we add an extra copy of this cycle to the above as well. We then add hairs to every element in each component of A' that is not M, such that the cardinality norm is the same as A, the maximum hair length is the same as A, but if we add any more hairs to any component other than M, one of these properties will fail to hold.

Now, we apply Theorem 3.3.2 to this modified version of A', with the multiplication coming from the component M, and call the resulting semigroup T. Two important things to note here is that when we multiply together any two element that are not in M, then we get something in the smallest cycle, and also that when we take $x \notin M$ and multiply it by $y \in M$, we get $a^k y$ for some $k \in \mathbf{N}$. These two facts will be important since we can remove any component (other than M and the smallest cycle) or remove any number of hairs from any component other than M (with the understanding that if one removes a hair, you must remove all of its predecessors as well), and still get a subsemigroup of the constructed semigroup T.

From here, we construct the semigroup as in Proposition 3.1.7 using the above defined semigroup T instead, treating the component M as being a different component type than any of the others (this is done so that if we have another component with a cycle of this type, then the only way we can multiply to get to an element in this cycle is to multiply this cycle by something in M, which allows us to remove hairs from this component to form a subsemigroup). The semigroup we get from this (call it S') now has the correct number of components and component types, but has too many hairs on every component other than M. However, as stated before, since we can remove any collection of hairs we like from the components other than M, so we can remove all the hairs that are not in A, and get that A arises as L_a for some a in the component M in this subsemigroup S of S'. This finishes the reverse direction, and completes the proof. □

In a similar way, one can also the analogous theorem for the infinite, non-separable case by requiring that the cardinality of the smallest cycle must be the same as the dimension of the Hilbert space.

The next case to consider is when the cycle decomposition has only $\mathbf{Z}$ cycles.

Theorem 3.3.9. *Let A be a basis map operator over an infinite dimensional, separable Hilbert space, with A having only $\mathbf{Z}$ cycles in its cycle decomposition. Then A can be realized as an L_a for a in some unital commutative, bounded semigroup S if and only if A is bounded and there is a $p \in \mathbf{N}$ such that every component of the graph of L_a has at most p elements in each grading equivalence class.*

Proof. We start with the forward direction. If A has only $\mathbf{Z}$ cycles, then we know from Lemma 3.3.1 that all the components of A must have a cycle part. Furthermore, we know from Theorem 3.2.1 that the component that $A = L_a$ comes from has all its hairs being at most k iterations of L_a away from the cycle part, for some $k \in \mathbf{N}$. Now, looking at any other component, if there are two elements c and d in it such that $a^k c \neq a^k d$ but

$a^{k+1}c = a^{k+1}d$, then we let f be an element in the cycle part of the component that L_a comes from, in the -1 grading class. Then $a^{k+1}cf = a^{k+1}df \Rightarrow (a^{k+1}f)c = (a^{k+1}f)d \Rightarrow a^k c = a^k d$, and so all the components of L_a must have hairs of length at most k away from the cycle part. Since A must be bounded, it must have a cardinality norm of at most some $m \in \mathbf{N}$, and so the grading classes of A must have at most m^k elements in them since every element in a particular grading class is a predecessor of the cycle element of that grading class times a^k (denote this by z). Because hair lengths are bounded by k, and the cardinality norm by m, z has at most m immediate predecessors, and each of those have at most m, and so on until we have at most m^k predecessors of z that are in the particular grading class. Thus, the number of elements in each grading class is bounded by m^k, and so they are uniformly bounded, which finishes the forward direction.

For the reverse direction, we pick a component that has hairs which are k iterations away from the cycle part, and let a be a hair that is of length $k - 1$, which also has a predecessor. Here now, we extend the basis map operator A to A' which includes new basis elements, such that every element has exactly l distinct predecessors (where l is the cardinality norm of A) except for the elements which are exactly k iterations away from the cycle part of their component, which have no predecessors, in a similar way as done in Theorem 3.3.8. From here, we use the same construction that was used in Theorem 3.3.8 for defining multiplication on the connected component of a, and call this connected semigroup S'. We note that the logic for showing that the products were associative before also holds here as well.

Next, we have two cases to consider: the case where A has finitely many connected components, and the case where it has infinitely many. In the first case, we let m be the number of connected components that A has, and we construct the semigroup $S' \oplus \mathbf{N}_m$, where $\mathbf{N}_m$ is the first m non-negative integers with the maximum as the multiplication. We can then form the desired semigroup S as a subsemigroup of this direct sum by first removing the basis elements of the form $(b, 0)$ from the connected component of A that were added to it in the construction of A', and then removing the elements (x, i) that do not appear in the graph of A (there is a lot of possible ways of doing this, but any way will work). This gives a semigroup with A arising as $L_{(a,0)}$.

For the second case of A having infinitely many connected components, we can just apply the same construction as in the above, but with $\mathbf{N} \cup \{0\}$, where the multiplication is just the usual addition on $\mathbf{N}$, and we remove elements in a similar way as in the finite number of components case.

Finally, we need to be sure that the multiplication defined above is actually bounded. In the case of a finite number of components, for fixed elements (w, z) and (x, y), if $(w, z)(c, d) = (x, y)$, then we must have that both $wc = x$ and $zd = y$. The maximum number of c's that can satisfy the first equation is l^k, where k is the maximum hair length of L_a in S', and l is the cardinality norm of A, while the maximum number of d's that can satisfy the second equation is m (the size of the second summand's semigroup). Thus the maximum cardinality norm for the left multiplications in this constructed semigroup S is $l^k m$, so S is not only bounded, but uniformly bounded. Lastly, in the case of there being infinitely many components, the only difference is in the second equation $zd = y$. Here, since the multiplication on the second summand is cancellative, the maximum cardinality norm for the left multiplications in this case is l^k, and in particular, is again uniformly bounded. Thus, we have indeed constructed the desired semigroup for the reverse implication, and so we're done. □

The above condition on the uniform finiteness of the equivalence classes of the basis map operator need not be required for the case of $\mathbf{N}$ cycles. However, in the cases that it does hold, an easy construction can be used to obtain a bounded semigroup with the desired basis map operator, if we also assume a uniform bound on the hair lengths as well.

Theorem 3.3.10. *Let A be a basis map operator over an infinite dimensional Hilbert space, with A having a full cycle decomposition, and only $\mathbf{N}$ cycles in this decomposition.*

Moreover, suppose that on each component of the graph of A, the length of every hair is at most some fixed number k. Then A can be realized as a L_a for a in some unital commutative, bounded semigroup S if and only if A is bounded.

Proof. We will construct a semigroup for which A arises as a L_a for some a in the semigroup by applying Theorem 3.3.9 to a modified basis map operator. We select any component of the graph of A which has a hair of length k, and for every other component we extend the multiplication by adding an infinite chain of predecessors onto the first element of the cycle. This turns each of the cycle parts of these components into $\mathbf{Z}$ cycles. Finally, for the component that was selected first, we look at all elements in this connected component, which are predecessors of $A^{k-1+i}a$, $i \in \mathbf{N}$, but are not predecessors of $A^{k-2}a$ (in terms of the graph, if one wished to find a path between a and these elements, one would have to go through the vertex of $A^{k-1}a$). Out of all of these elements, we pick one in the lowest grading class, and then add an infinite branch of hairs going into it. This makes a a hair of length $l-1$ of this new cycle, where $l \in \mathbf{N}$ and $l \geq k$.

From here, we take the component of a, and form the basis map operator A' with the property that every element that is not a hair of length l has exactly m distinct predecessors, where m is the cardinality norm of A, while the hairs of length l have no predecessors at all. From here we note that each grading class of A has at most m^l elements, so we can construct the semigroup S' from Theorem 3.3.9. This makes a huge, bounded semigroup, in which L_a restricted to some subset of the basis is A, but we need to be sure that when we restrict to such a basis, the multiplication is still closed.

First, we note that for the component of a, since we chose to add the infinite branch of hairs to an element in the lowest possible grading class, if we remove the infinite branch of hairs, and the extra elements of A' that we added, then we get a closed subsemigroup on this component. This is true since if we multiply something that is a predecessor of $A^{k-2}a$ (in Theorem 3.2.1's notation, these elements were the complement of set N) by x in this component, then the result is just x times a power of a (which is closed in this subsemigroup), and if we multiply something in N by something else in N, then the result is an element in the cycle whose grading is the sum of the gradings of these two elements (which again is closed since the grading classes of N are all non-negative, and for every element in N, there is an element in what remains of the old $\mathbf{Z}$ cycle which has the same grading as it). This makes multiplication closed on this component.

For each of the rest of the components, we can remove the infinite branch of hairs that we added to them and still have that the multiplication is closed since these elements all have negative grading classes on their respective components. Then, as in Theorem 3.3.9, we can also remove the added hairs in these components for the same reason that we could remove them in the connected component of a: $xy = a^k y$ if x is not in N, while xy is in the cycle (which for these components has all the non-negative grading classes) if both x and y are in N.

Finally, the forward direction of the implication is obvious. □

Using the above, with a slight bit of work, we can form a corollary that includes the case of both $\mathbf{N}$ and $\mathbf{Z}$ cycles.

Corollary 3.3.4. *Let A be a basis map operator over an infinite dimensional Hilbert space, with A having a full cycle decomposition, and only $\mathbf{N}$ and $\mathbf{Z}$ cycles in this decomposition. Moreover, suppose that on each component of the graph of A, the length of every hair is at most some fixed number k. Then A can be realized as a L_a for a in some unital commutative, bounded semigroup S if and only if A is bounded, and it is not the case that A has infinitely many $\mathbf{N}$ cycles and finitely many $\mathbf{Z}$ cycles.*

Proof. For the forward direction, we apply the same construction as in the case of Theorem 3.3.10 except we separate the components by their cycle types. Then, we apply the same idea as before, but when we form the direct sum semigroup, we include an additional factor of $\mathbf{N}_2$. Then, the components of type $\mathbf{N}$ are grouped into the components corresponding to $(S, n, 0)$, while the $\mathbf{Z}$ components are grouped into the components corresponding to $(S, n, 1)$. When we have a bijection between the set of $\mathbf{N}$ cycles and the set of $\mathbf{Z}$ cycles, we can just take the subsemigroup of this semigroup, where we remove the negative grading elements of the form $(S, n, 0)$ (like in Theorem 3.3.10), while keeping the rest of the form $(S, n, 1)$. Here, we obtain A as $L_{(a,0,0)}$.

If there is no bijection between the semigroups, then we apply the idea of Proposition 3.1.7 and take an appropriate subsemigroup. That is, for infinitely many $\mathbf{Z}$ cycles and finitely many $\mathbf{N}$ cycles (say k of them), we take the semigroup $S \oplus \mathbf{N}_\infty \oplus \mathbf{N}_2$, where $\mathbf{N}_\infty$ refers to the non-negative integers with maximum as the multiplication. We associate the $\mathbf{N}$ cycles with elements of the form $(S, n, 0)$, where $n < k$, and we associate the $\mathbf{Z}$ cycles with elements of the form $(S, m, 1)$, where m can be any non-negative integer. From here, we take above semigroup, remove all the elements that are not of the above form, and then we remove elements from each cycle component so as to get $L_{(a,0,0)} = A$. For the case of both $\mathbf{N}$ and $\mathbf{Z}$ cycles being finite in number, we can let m be the number of $\mathbf{N}$ cycles, and n the number of $\mathbf{Z}$ cycles, and take the semigroup as $S \oplus \mathbf{N}_{m+n} \oplus \mathbf{N}_2$. Here we associate the $\mathbf{N}$ cycles with the elements $(S, i, 0)$, $i < m$, and the $\mathbf{Z}$ cycles with the elements $(S, j, 1)$, $j \geq m$. Taking the same subsemigroup then gives A as $L_{(a,0,0)}$.

Finally, boundedness follows from the fact that the above semigroups are subsemigroups of a bounded semigroup (the direct sum of S along with $\mathbf{N}_m$, $\mathbf{N}_\infty$, etc.). □

The case of basis map operators consisting of $\mathbf{N}$ cycles without a uniform bound on their norms is rather difficult since the cycle structure of the components do not in general form a nice ideal like in the other cases, and makes the method of taking appropriate subsemigroups of a direct sum of semigroups not as feasible as it would have to be to obtain the desired classification result in this case. However, some beginning progress can be made.

Proposition 3.3.1. *Supposed that A is a bounded basis map operator arising as L_a in some bounded, commutative semigroup S. Suppose furthermore that the graph of A contains components that have uniform boundedness on the elements in the grading* *equivalence classes while other components do not. Then a must be on a component that does have the uniform boundedness of the elements in the grading equivalence classes.*

Proof. Suppose otherwise, then a is on a component without uniform boundedness of the element in the grading classes. Let c be an element on a component with the boundedness condition (moreover, let this bound be denoted l), and let $b_0, b_1, ..., b_k$ be a collection of elements in the component of a that are in same grading class. Then $cb_0, cb_1, ..., cb_k$ are all in the same grading class in the component of c, and thus, the cardinality norm of L_c must be at least k/l (L_c sends this set of k elements to a set of l elements, so there must be at least k/l elements that are sent to the same element under the map). However, since we can pick any number of elements in the same grading class in the connected component of a, we have that k can be chosen to be as big as we like. Thus, L_c is unbounded, and we are done. □

3.4 Applications

We have spent considerable time and effort trying to classify what a semigroup operator looks like, but this is a somewhat local property and does not immediately tell us that much about what the rest semigroup can be itself. However, as we have seen, there is some global information attached to such a semigroup operator, as well as some restrictions as to which basis element the multiplication comes from.

One example of an application of our results is in the area of extensions of partially defined semigroupoids to semigroups. In general, this problem is rather difficult, but for some cases (like the cases we have dealt with involving a single row being defined) the problem can be solved explicitly. We can also extend the results to include the case of allowing extra elements to be added to the multiplication table. We call this the Function Extension Theorem.

Theorem 3.4.1. *(Function Extension Theorem) Given a set B and a map $f : B \to B$,* *there exists a semigroup B', and $f' : B' \to B'$, such that $B \subset B'$, $f'|_B = f$, and f' is given by left multiplication of some element in B'. Moreover, the semigroup can be chosen so that it is commutative.*

Proof. We look at the structure of f as a basis map operator on the set B. We add new elements c_i, $i \in \mathbf{N}$ and d, to B (forming a set B') and extend the definition of f to a f' defined on B' by $f(c_i) = c_{i+1}$ and $f(d) = d$. In terms of the graph structure of f, this process just adds a new component consisting of an $\mathbf{N}$ cycle, and a new component consisting of a 1-cycle. Since f' as a basis map operator now has an $\mathbf{N}$ cycle, and a cycle whose order divides the orders of all the other cycles (the 1-cycle), By Theorem 3.3.7, there exists a semigroup multiplication on B' such that f' is left multiplication by some element in B' (specifically in this case, it will be c_1). Moreover, this semigroup is also commutative. □

We note that there is no corresponding theorem for extensions to bounded semigroups. For example, as we have seen, no bounded semigroups can contain multiplications that have infinite and finite cycles in them, or finite cycles without a least common multiple of their orders. So the corresponding theorem for the above is dependent on some conditions. We outline some broader cases.

Theorem 3.4.2. *(Bounded Function Extension Theorem For Finite Cycles) Given a countable set B and a map $f : B \to B$ with only finite cycles in its cycle decomposition, there exists a bounded semigroup B', and $f' : B' \to B'$, such that $B \subset B'$, $f'|_B = f$, and f' is given by left multiplication of some element in B' if and only if the cycles in the cycle decomposition of the basis map operator of f have a bound on the orders of the cycles, there is a bound for the maximum hair length on these cycles, and as a basis map operator, f is bounded. Moreover, the semigroup can be chosen so that it is commutative.*

Proof. For reverse direction, since the orders of the cycles is bounded by some integer k, we include a new cycle of order $k!$ to f. Next, we add a branch of hairs of length l

to this cycle, where l is the bound on the hair lengths for the other cycles. Next, we include a countable number of cycles of order 1. By Theorem 3.3.8, we have that this multiplication can be extended to the entire semigroup.

For the forward direction, we note that if there is no bound on the orders of the finite cycles, then we can not form a finite cycle whose order is a multiple of all the others, and so if we could extend the function, we would have to use an infinite cycle, and thus, the semigroup can not be bounded. Likewise, if the maximum hair lengths do not have a bound, then one of the cycles whose order is a multiple of all the other orders will have to contain an infinite branch of hairs, and thus, it would have to again contain an infinite cycle, making the extension unbounded. Finally, if the basis map operator of f is not bounded, then it obviously can't be extended to a bounded semigroup. □

In a more complicated way, some other cases can also be handled.

Theorem 3.4.3. *(Uniformly Bounded Function Extension Theorem For* $\mathbf{N}$ *Cycles) Let* B *be a countable set and* f *a map from* B *to* B*, which as a basis map operator has only* $\mathbf{N}$ *cycles in its cycle decomposition. There exists a bounded (commutative) semigroup* B'*, and* $f' : B' \to B'$*, such that* $B \subset B'$*,* $f'|_B = f$*, and* f' *is given by left multiplication of some element in* B' *if and only if* f*, as a basis map operator, is bounded.*

Proof. The forward direction is obvious, so we need only prove the reverse. Since f is bounded, it has a cardinality norm of, say, $k \in \mathbf{N}$. Using this k, we extend f so that for every element z in its new domain which is not in the lowest grading class of its cycle, there are exactly k elements y such that $f(y) = z$ (that is to say, every basis element has exactly k predecessors under f except those that are the in the 0-grading class of its connected component). As well, we extend the multiplication to include countably many connected components.

By Theorem 3.2.1, we know that if there was only one component, then this extended basis map operator would be possible as a semigroup operator coming from a commutative semigroup S'. To account for infinitely many components though, all we have to do is take $S' \oplus \mathbf{Z}$, and we have that $f' = L_{(a,0)}$, where a is the element that the multiplication arises from in Theorem 3.2.1.

□

For the case of there being both $\mathbf{N}$ and $\mathbf{Z}$ cycles however (or without a loss of generality, just $\mathbf{Z}$ cycles), the extensions to take do not seem as obvious.

Another application of the ideas from these sections is to extensions of semigroup actions.

Theorem 3.4.4. *(Semigroup Action Extension Theorem) Let S be a semigroup which has an action $\phi(.,.) : S \times B \mapsto B$ on a set B. Then there exists a semigroup T such that $T = S \cup B$, S is a subsemigroup of T and for $s \in S$, $b \in B$, $sb = \phi(s, b)$. Moreover, if the action has a fixed point, and S is commutative, then T can be made to be commutative.*

Proof. For this proof, we note that the semigroup S and the action ϕ determines the multiplication for L_s, $s \in S \subset S \cup B$. Like in the non-commutative cases, since we have associativity for all composable products, we can define L_b, $b \in B$ as $L_b(x) = b$ for all $x \in S \cup B$, and obtain an associative multiplication on all of $S \cup B$.

In the case that there is a fixed point of the action, we can apply the commutative version of the construction where for $s \in S$, $b \in B$, $bs = sb = \phi(s, b)$, and for any $b, c \in B$, $bc = d$, where d is the fixed point of the action. For the same reason as in the previous theorems, this gives an associative multiplication on $S \cup B$, which is also commutative. □

Finally, we end this section with one last application to finitely generated, commutative, connected, and bounded semigroups. First however, we need a lemma.

Lemma 3.4.1. *Let S be a bounded, connected semigroup with L_a being connected. Denote $Dist(c, a)$ as the length of the hair $c \in S$ to the cycle induced by a (if c is in the cycle, then* *we will define* $Dist(c, a)$ *as* 0*). Then for* $c, d \in S$*,* $Dist(cd, a) \leq \max Dist(c, b), Dist(d, a)$*.*

Proof. Let $Dist(c, a) = k$ and $Dist(d, a) = m$. Then $a^k c = a^{k+i}$, and $a^m d = a^{m+j}$ for some $i, j \geq 0$. Thus, for some $i, j \geq 0$, $(a^{\max k,m} c)d = a^{\max k,m+i} d = a^{\max k,m+i+j}$, thus $Dist(cd, a) \leq \max Dist(c, a), Dist(d, a)$. □

This notion of distance to the cycle induced by a gives us a new method for determining structure properties of connected subsemigroups of a semigroup that is in a sense transverse to the grading functor (the grading tells us which element in the cycle the element corresponds to, while the distance tells us how far it is from the cycle).

With this, we are now able to show that for finitely generated commutative and connected semigroups, boundedness is the same as uniform boundedness.

Theorem 3.4.5. *Let* S *be an commutative, connected, unital, bounded semigroup. If* S *is finitely generated, then it is uniformly bounded.*

Proof. To prove this result, by commutativity and S being finitely generated, we need only show that for any element a in the semigroup, $\sup_{n \in \mathbf{N}} ||a^n||$ is bounded. If L_a has finite hairs, then we know that there must be a maximum length m to these hairs. Since L_a is bounded, this means that L_a^m is bounded and has only hairs of length 1. Multiplying by L_a again only shifts the hairs around the cycle part.

Now we have to investigate the cycle part. If L_a has a nice cycle part, say just one finite cycle, or just one shift, then as we multiply by L_a, the cycle parts may increase (for example, like the unilateral shift and its square), but the norms do not since as we recall from the first proposition that the norm of L_a is determined by the maximum number of arrows going to a particular point, and if the graph splits, then this cardinality norm doesn't change.

However, there is a possible problem if the cycle part is an infinite shift and there are an infinite number of hairs. For example, one could imagine a Y shaped graph, and then split each leg of the Y into two new Y shapes and so on. As we continue to take powers of this operator, the norm will increase.

We wish to show that if a semigroup does indeed have such an operator, then it must be infinitely generated anyway. From commutativity, we know that the only case that this can happen is when the cycle is of type $\mathbf{N}$ and the lengths of the hairs are not bounded (for example the multiplication given by Figure 3.4).

Suppose that the hair lengths are unbounded and that the semigroup is finitely generated. Then for some elements, $a_0, ..., a_k \in S$, the semigroup generated by them is the entire semigroup. However, each one of these must be a hair of finite length, and by Lemma 3.4.1, all their products will always be a bounded distance away from the cycle part (concretely, the maximum of $\{Dist(a_0, a), ..., Dist(a_k, a)\}$, where a is the multiplication that the graph arises from). Hence, we have a contradiction, and so S in this case must be infinitely generated, and thus, a finitely generated, commutative, and connected semigroup that is bounded must also be uniformly bounded. □

We note however that not all finitely generated commutative and connected semigroups are uniformly bounded. The boundedness condition is required in the above since one could take any infinite but finitely generated semigroup and add a zero element z (that is, $zg = gz = z$ for any generator g). This makes the semigroup connected by L_z, but L_z is unbounded since it sends everything to itself.

As well, we note that the requirement that S be commutative is required. As a counter example, take S to be the bicyclic semigroup $< a, h \mid ah = e >$. Then this semigroup is bounded, connected, and finitely generated, but the cardinality norms of a^n tend to infinity as n tends to infinity.

Chapter 4

Basis Map Operator Properties

Before continuing on, it will be useful to first explore a little more about the semigroup operators that were discussed in the previous chapter, but without having to refer to the underlying semigroup. In this setting, it often more useful to work with basis map operators instead of just semigroup ones.

As defined earlier, a basis map operator is an operator in $B(H)$ that sends a basis element to another basis element (equivalently, its matrix has exactly one 1 in each column and zeros elsewhere). Each $L_s \subset L(S)$ for $s \in S$ is a basis map operator on $l^2(S)$, but as seen before, basis map operators are more general than what the operators L_s can be.

In this section we will explore some of the properties of basis map operators that will be useful for the following chapters. In particular, we will be concerned with the Jordan Canonical Form, as well as diagonalizability conditions for basis map operators in finite dimensions.

As with any computation of the Jordan Canonical Form, we will have to find its eigenvalues.

Lemma 4.0.2. *Let W be a basis map operator over a finite dimensional Hilbert space, then the eigenvalues of W will be the union of the a_n-th roots of unity for some finite* *collection of* $a_n \in \mathbf{N}$*, and if* W *is not a permutation, then also* 0*.*

Proof. Since W is a finite dimensional basis map operator, it has a cycle decomposition, so we can take all of the elements in the cycles of W and permute the basis of the basis map operator so that these k elements are at the beginning of the basis. Next, we permute each of the hairs of length one (if there are any) so that they are next in the basis. Likewise, continue this for the hairs of length two and so on.

Now, since the the first $k \times k$ block of this permuted basis map matrix is a permutation, we can diagonalize that part of it. This now gives us an upper triangular matrix, with diagonal entries consisting of the a_n-th roots of unity for some collection of $a_n \in \mathbf{N}$ (comming from the permutation), and 0 for the rest of the diagonal entries since the columns for the hairs of length one have their ones above the diagonal (the images of these hairs are in the cycle), and likewise for the hairs of length two (their images are the hairs of length one, which are before them in the reordered basis), and so on.

Thus, if W is just a permutation, then the eigenvalues will be a union of the a_n-th roots of unity for some collection of $a_n \in \mathbf{N}$, and if it is not a permutation it will also include 0. □

Theorem 4.0.6. *(Jordan Canonical Form For Basis Map Operators) If* W *is a basis map operator for a finite dimensional Hilbert space, then the Jordan Canonical Form of* W *consists of Jordan* 1*-blocks with eigenvalues as a union of the* a_n*-th roots of unity for some collection of* $a_n \in \mathbf{N}$*, and the Jordan blocks for the* 0 *eigenvalue. The longest chain of hairs (say, of length* l*) going into some element of the cycle gives a Jordan block of size* l*, and then the next longest chain of elements (say, of length* j*) going into the cycle or some basis element of the previous chain gives another Jordan block of size* j*, and then the next longest chain of elements (say, of length* k*) going into the cycle or some basis element of the previous chains gives another Jordan block of size* k*, and so on.*

Proof. To prove this, we will give an explicit basis that will put W into its Jordan Canonical Form. First, we take W and permute the basis so that the cycles come first.

From Lemma 4.0.2, we know that these basis elements will give us the desired eigenvalues for W with an appropriate change of basis on the cycle elements.

For the rest of the operator W, we take a hair a of maximal length going into the cycle, and do a change of basis to send it to $a - b$, where b is an element in the hair's cycle whose grading class is the same as a. Likewise we do same for $W(a)$, sending it to $W(a) - W(b)$, and so on for all $W^k(a)$'s that are not in the cycle. Since for some $l \in \mathbf{N}$, $W^l(a)$ will be in the cycle, and $W^l(b)$ will have the same grading class as $W^l(a)$, $W^l(a) - W^l(b)$ will equal zero, and so once we have done the above change of basis, we have that W restricted to these elements $W^k(a) - W^k(b)$ form a Jordan block of size l. We denote set of basis elements consisting of the cycle, and a along with powers of W applied to it as the set M.

Next, of the basis elements that are not in M, we take an element c that is furthest away from it (that is, some element such that for some $j \in \mathbf{N}$, $W^j(c) \in M$, $W^{j-1}(c) \notin M$, and of all elements not in M, this j is maximal). We do a change of basis to take c to $c-d$, where $d \in M$ with the property that for the smallest $j \in \mathbf{N}$, such that $W^j(c) \in M$, $W^j(c) = W^j(d)$. Again, as before, W restricted to $c-d$ along with $W^k(c) - W^k(d)$ form another Jordan block of size j.

We continue the above process inductively until we have done a change of basis to the entire basis. This gives a change of basis of W consisting of diagonal entries and Jordan blocks, making it the Jordan canonical form of W. □

Diagonalizability of the left multiplications can also be easily characterized as follows. While the following proposition is phrased for semigroups, we note that with the exception of 3), it also holds for basis map operators.

Proposition 4.0.1. *For a bounded, finite semigroup S, $s \in S$, the following are equivalent:*

1.) L_s is diagonalizable.

2.) The graph of L_s has hairs of length at most 1.

3.) For all $t, d \in S$, if $s^2t = s^2d$ then $st = sd$.

4.) For all $v \in l_2(S)$, if $L_s^2(v) = 0$ then $L_s(v) = 0$.

Proof. 1) $\Rightarrow$ 2): By Theorem 4.0.6, we know that if L_s is diagonalizable, then it must not have any hairs of length 2. Thus, it must have hairs of length at most one.

2) $\Rightarrow$ 3): Since all the hairs of L_s are of length one, this means that sS is the set of cycles of L_s. Left multiplication of L_s with respect to this set is a permutation, and so if $s(sd) = s(st)$ then $sd = st$.

3) $\Rightarrow$ 4): Let $v = c_0e_0 + c_1e_1 + ... + c_ne_n$, where $c_i \in \mathbf{C}$ and $e_i \in S$, then $L_s^2(v) = c_0L_s^2(e_0) + c_1L_s^2(e_1) + ... + c_nL_s^2(e_n) = 0$. Breaking this up into coordinates gives us a bunch of equations of the form $a_0L_s^2(e_{b_0}) + a_1L_s^2(e_{b_1}) + ... + a_kL_s^2(e_{b_k}) = 0$, where $L_s^2(e_{b_1}) = L_s^2(e_{b_2}) = ... = L_s^2(e_{b_n})$. Since $L_s^2(e_{b_i}) = L_s^2(e_{b_j})$, we get that $L_s(e_{b_i}) = L_s(e_{b_j})$, and so $L_s(v)$ must be 0 as well.

4) $\Rightarrow$ 1): Switch L_s into its Jordan Canonical Form. Then all Jordan blocks will be of size one, since otherwise the first entry in such a Jordan block will not satisfy 4). Thus L_s is diagonalizable. □

Finally, we remark on a decomposition of basis map operators that is similar in nature to the Wold decomposition of isometries. It should be noted though that unlike in the isometry case, we can not put the operator into such a decomposition with just a unitary change of basis.

Theorem 4.0.7. *(Wold Decomposition For Basis Map Operators) Let A be a bounded basis map operator. Then A is similar to some direct sum of U, T and S, where U is unitary, T is a direct sum of unilateral shifts and S is a direct sum of the adjoint of the unilateral shift and finite rank Jordan blocks with eigenvalues $\lambda = 0$, if and only if, the cardinality norms of the positive powers of A are uniformly bounded.*

Proof. We show the forward direction first and assume that a basis map operator A has such a decomposition. If we suppose otherwise that the powers of A are not uniformly bounded, then we note that under the change of basis to the Wold decomposition, we have that A is uniformly bounded (since it is a direct sum of unitares, isometries, Jordan blocks (with 0 along their diagonals), and adjoints of these), giving us a contradiction.

So we just have to show the reverse direction, where we assume that A is uniformly bounded, and find the desired decomposition. First, we note that A must have a full cycle decomposition. For, if we assumed otherwise, then some component of the graph of A would not have a cycle, and by Proposition 3.2.2, for this component, while the grading classes have $\mathbf{Z}$ type, every element would only have finitely many predecessors. Since A has a uniform bound on the number of elements in each grading class, we pick a grading class M, and note that for some sufficiently large power k of A, all of these elements in this grading class will be sent to the same element (call this element b). However, if this component is of $\mathbf{Z}$ type, then this component would have to have infinitely many grading classes before the grading class M. If each of these grading classes had at least one element in them, then each of these element will eventually be sent into one of the element in M, and so they all would have to be predecessors of b, contradicting the fact that b must have only finitely many predecessors.

Now that we know that A must have a full cycle decomposition, we can find the desired unitary U. Using the cycle decomposition on A, and note that for each cycle that A has which is of finite or $\mathbf{Z}$ type, then A restricted to this cycle is unitary. The unitary U then becomes the restriction of A to the union of these cycles. Then, applying the cycle part construction, we can take A restricted to the $\mathbf{N}$ cycles to get a direct sum of unilateral shifts T.

These two direct summands have been unilaterally chosen, but for the S, we must use a non-unitary change of basis. First, we restrict our attention to a single component of A (call this component N). Here, we use a change of basis where we send the cycle to itself, and for all the other elements x in this component, we send x to $x - y$, where y is the element in the cycle with the same grading as x. After we've done this change of basis to a new operator (call it A_0), the graph of this part of the operator now has a bunch of connected components, but instead of just being a basis map operator, it can send some basis elements to the zero vector. The important thing about this change of basis however, is that the number of elements in each grading class of the components is now strictly less than before. This is because, if two elements (say, w and z) in A_0's graph are in the same grading class (that is, for some $k \in \mathbf{N}$, $A_0^k(w) = A_0^k(z)$), then their preimages under the change of basis would have to have been in the same grading class under A, since $A_0^k(w) = A_0^k(z)$ means that $A^k(w_0 - c) = A^k(z_0 - d)$ for some c and d in the cycle of the component N, with c having the same grading as w_0 and d having the same grading as z_0. This however, means that w_0 and z_0 must have the same grading under A.

The important thing about this change of basis however, is that the new operator A_0 now has a cardinality norm (where we extend our previous definition of the cardinality norm of a basis map operator to include the possibility that the operator can send some elements to the zero vector), of at most $k-1$, where k is the cardinality norm of A. This is because from the above argument we know that if two basis elements of A_0 are in the same grading class under A_0, then their preimages under the change of basis must be in the same grading class of A, (making the cardinality norm of A_0 at most k), but we also know that the cycle of A that we chose is in a separate component after the change of basis (since all the hairs are sent to a difference of the original basis elements, and so under enough iterations of A, they go to the zero vector and never into the cycle itself). Thus, the cardinality norm of A_0 must be at most $k - 1$.

After doing the, we have to choose a cycle for each connected component of the graph of A_0, noting that it is the same procedure as with basis map operators, except the cycle can eventually terminate and go to the zero vector. After choosing a cycle, we can do another change of basis with the same idea of sending x to $x - y$, and we further reduce the cardinality norm of the result by 1, and we can inductively keep doing this until we have an operator with a cardinality norm of 1, at which point, since it is a basis map operator (with the exception that it can send some basis elements to the zero vector), it must be a direct sum of finite cycles, **N** cycles, and **Z** cycles, except that some of these cycles will terminate at some stage, and thus either be an infinite branch of elements going that eventually goes to the zero vector (and thus the operator restricted to this component is the adjoint of the unilateral shift) or is a finite branch of n elements that goes to the zero vector (and thus the restriction is a Jorden block of size n with an eigenvalue of 0).

Thus, after we compose these $k-1$ changes of basis together, we have the desired change of basis for the restriction of A to the component N. To finish the decomposition, we proceed as follows. First, we take the basis elements of A, and form the equivalence relation based on connectedness ($x\ y$ if for some $n, m \in \mathbf{N}$, $A^n x = A^m y$). We then use the axiom of choice to pick a representative for each equivalence class, and form the connected component for each representative, and decompose the operator A into a direct sum of its cycle parts to these connected components, as done in Theorem 3.1.1. We then choose a cycle for each of these components (requiring again, the axiom of choice), we preform the algorithm above to each component and combine all the change of bases on the individual components into a single change of basis of the entire Hilbert space (each change of basis was defined only on a direct summand of A). After applying this algorithm $k-1$ times (noting that at each step we have to use the axiom of choice again to form the new connected components of the operator and to choose a new cycle for each such component), we have that doing these $k-1$ change of bases, and again, using the axiom of choice, we can form the connected components of the final operator, in which every connected component is either a cycle, or a cycle that eventually terminates. Moreover, and we also have a change of basis that sends A to a direct sum of cyclic permutations (finite cycles), bilateral shifts (**Z** cycles), unilateral shifts (**N** cycles), adjoints of the unilateral shift (**Z** cycles that eventually terminate and go to zero), and Jordan blocks with eigenvalues of 0 (finite or $\mathbf{N}$ cycles that eventually terminate and go to zero).

Finally, grouping the cyclic permutations and bilateral shift summands into a single operator, we obtain the unitary U, grouping the unilateral shift summands together gives us T, and grouping the adjoint of the unilateral shift and Jordan block summands together gives us S. Thus, we have that A is similar to a direct sum of U, T, and S. $\square$

We note, as implied by the proof of the result, that the corresponding decomposition also holds for modified basis map operators that may send some basis elements to zero, granted that the powers of these operators are uniformly bounded in norms.

Chapter 5

Reflexivity

Davidson and Pitts have done much work on the free semigroup of n generators, and in particular have shown that they are in fact hyper reflexive [1]. One would desire similar properties in the non-free case, or that the very least instead of hyper reflexivity just reflexivity.

However, when one does a quick investigation, one sees that these semigroup algebras may not be reflexive. As an example, take the simplest, non-trivial, unital semigroup that contains a multiplication with a hair of length 2. This semigroup has the following multiplication table:

.	a	b	c
a	a	a	a
b	a	a	b
c	a	b	c

Looking at $L(S)$, we obtain the algebra

$$\begin{pmatrix} c+d+e & c+d & c \\ 0 & e & d \\ 0 & 0 & e \end{pmatrix} \text{ with } c, d, e \in \mathbf{C}.$$

This subalgebra of the 3×3 matrices, shares same invariant subspaces of the matrix that has zeros everywhere except a one in the middle of the first row and a minus one in the middle of the second row. Hence, it is not reflexive. Another, less explicit approach as to why this algebra is not reflexive is because S is a singly generated semigroup with identity, so reflexivity for $L(S)$ is just reflexivity for the generator L_b. A quick computation of L_b's Jordan Canonical Form, along with Deddens and Fillmore's characterization of reflexive linear transformations on finite dimensional Hilbert spaces in [2], tells us that $L(S)$ can not be reflexive.

Using this, we can expand the above result to include all singly generated semigroups. However, we first should recall what a singly generated semigroup looks like.

Proposition 5.0.2. *If S is a singly generated unital semigroup with generator a, then either S is $\mathbf{N} \cup \{0\}$, or for some $n \in \mathbf{N} \cup \{0\}$, L_a is a finite cycle with a single branch of hairs of length n attached. Likewise, if L_a is a finite cycle with a single branch of hairs of finite length attached, then the semigroup S it arises from must be singly generated.*

Proof. In the infinite case, the powers of a give a semigroup that is the same as the natural numbers, while the identity acts as 0.

In the finite case, for some $k \in \mathbf{N}$, $a^k = a^l$ for some $l \in \mathbf{N}$, $l < k$. Including the identity, we get a cycle (if $l = 0$), or a cycle (of length $k - l$) with a branch of hairs (of length l) attached. For the converse, we know from the structure theorems of the previous section that a has to be the second furthest element on the hair, and so it with the identity generates the entire semigroup. □

Now, we can answer when such semigroups have a reflexive left regular representation.

Proposition 5.0.3. *If S is a finite, singly generated, unital semigroup then $L(S)$ is reflexive if and only if $S = G$ or $S = G \cup e$, where G is a cyclic group and e is a new identity adjoined on.*

Proof. We first assume that $L(S)$ is reflexive. We use the previous proposition with Deddens and Fillmore's result in [2], and note that the generator L_a will have a single Jordan block for the eigenvalue $\lambda = 0$. If the hair is of length more than one, then the Jordan block will be of size two or more, making L_a not reflexive. Thus, L_a only hairs of length at most one, moreover, it must have at most one such hair (otherwise the semigroup would not be singly generated). If it has a hair, then it must be the identity, while restricting to the cycle gives a cyclic group.

For the converse, the generator of S will be a cycle with at most one hair sticking out of it. In the Jordan Canonical Form this will be diagonal, and thus a reflexive transformation. Since $L(S)$ is singly generated, this gives that $L(S)$ is reflexive. □

For the above case, we see that the process of adding on a new identity to a group (whose left regular representation is reflexive) will result in a semigroup S in which $L(S)$ is still reflexive. This result however is true for any semigroup.

Theorem 5.0.8. *Let S be a bounded, unital semigroup with $L(S)$ being reflexive. If $S' = S \cup \{e\}$, where e is a new adjoined identity, then $L(S')$ is also reflexive.*

Proof. We begin by noting that the invariant subspaces of $L(S)$ (with understanding of these as $l^2(S) \subset l^2(S')$) are included in the lattice of $L(S')$ since they are all invariant subspace of the new multiplication (the new identity) as well. Thus, we may think of any matrix T in $Alg(Lat(L(S')))$ as follows:

$$T \in Alg(Lat(L(S'))) = \begin{pmatrix} & B & & x \\ 0 & 0 & \ldots & y \end{pmatrix}$$

where $B \in L(S)$, x is some n-dimensional vector (n the cardinality of S), and y some complex number. Here however, we note that, $i - e$, where i was the old identity of S, and e the new identity, is an invariant subspace for $L(S')$ (it is in the null space for any L_a, where $a \in S$, and $L_e(i - e) = i - e$. This determines the right column of T since

$T(i-e) = \lambda(i-e) \to (T-\lambda I)(i-e) = 0 \to (T-\lambda I)(i) = (T-\lambda I)(e)$. Since $I = L_e$, all we have to show now is that $(T - \lambda I)$ is in $L(S)$ when considered as a subset of $L(S')$.

Since $(T-\lambda I)(i) = (T-\lambda I)(e)$, $\langle (T-\lambda I)e, e\rangle = 0$, and so we may think of $(T-\lambda I)$ as:

$$(T-\lambda I) \in Alg(Lat(L(S'))) = \begin{pmatrix} & B' & & x \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

where $B' = B - \lambda L_i \in L(S)$, and $x = B'(i)$. However, since for any $a \in S$, L_a as an element of $L(S')$, has the property that $L_a(i) = L_a(e)$, and taking linear combinations, this is true for any $y \in L(S)$ as well. This precisely means that the column of e is the same as the column of i. Thus, since $x = B'(i)$, and $B' \in L(S)$, we have that $(T - \lambda I)$ is the same linear combination of L_a's as B', but embedded in $L(S')$. Thus $(T - \lambda I) \in L(S')$, and so $T \in L(S')$. Since T was an arbitrary matrix in $Alg(Lat(L(S')))$, we have that $L(S')$ must be reflexive, as desired.

□

From this, we also have a nice corollary.

Corollary 5.0.1. *The algebras $L(N_i)$, where N_i is the semigroup of integers from 0 to i with $ab = \max\{a, b\}$, are reflexive.*

Proof. We note that N_0 is the group of one element and so $L(N_0)$ is reflexive, while N_i is isomorphic to N_{i-1} with a new identity adjoined on. So by induction, all the $L(N_i)$'s are reflexive. □

Since the non-diagonalizability of a basis map operator comes from its possession of hairs of length greater than 1, one may be interested in looking at semigroups that have sufficiently small hairs for their multiplications. In particular, Proposition 5.0.3 seems to indicate that semigroups with too many hairs of length more than one will not have reflexive left regular representations.

As a starting point, one may be interested in semigroups that have no hairs, such as left cancellative semigroups. For a finite semigroup though, this implies that each L_s is actually a unitary, which in turn implies that L_s^{-1} also comes from a semigroup element, which in turn means that $L(S)$ is *-closed. So $L(S)$ is a von Neumann algebra, and moreover, S is actually a group.

If one wished to study locally diagonalizable semigroups (that is, each L_s is diagonalizable), then we see that in the finite case, each one will turn out to be a regular semigroup. These semigroups have the property that for each $x \in S$, there is a $y \in S$ such that $xyx = x$ and $yxy = y$. Unlike inverse semigroups though, regular semigroups need not have a unique y for each x.

Proposition 5.0.4. *If S is a finite, locally diagonalizable unital semigroup, then S is a regular semigroup.*

Proof. In the finite case, each L_s being diagonalizable implies that it has hairs of length at most one, and since S has a unit, this means that the s for L_s must be on a cycle part. Thus $s^k = s$ for some $k > 1$, and so $ss^ls = s$ for some l. □

And interesting question to ask now is if the locally diagonalizable condition is equivalent to the regular condition for semigroups in the finite case.

With the more general condition of a twice cancellative semigroup, then in case of an infinite semigroup, this need not be true. In particular, cancellative implies twice cancellative, and so taking the natural numbers with 0 along with addition gives a semigroup that is twice cancellative but not regular.

Chapter 6

The Universal Map Algebra

6.1 Definition and Properties

After having studied the structure of the induced left multiplication operators in $L(S)$, one may be interested in constructing a universal object for all possible algebras $L(S)$.

Definition 6.1.1. *For a given dimension n, the Universal Map Algebra, which we shall denote as $UMA(n)$ or just UMA where the dimension is understood from context, is the algebra generated by all basis map operators in $B(H)$. Moreover, since the collection of basis map operators is closed under multiplication, the Universal Map Algebra can be defined as just the span of all basis map operators in $B(H)$.*

It should be noted here that UMA is not defined to be closed in any topology, but its closure will be consider later.

The Universal Map Algebra may be considered as being a universal object in the same sense as the group S_n can be considered as a universal object for all groups of size n or less. Like in the group case, all semigroups of size n (and consequently, all semigroups with fewer than than n elements) can be realized as a subsemigroup of the basis map operators in UMA. In this way, one can see that all the left regular representations of semigroups of size n sit inside $UMA(n)$.

In finite dimensions, the vector space dimension of $UMA(n)$ can be easily computed.

Proposition 6.1.1. *For $n \in \mathbf{N}$, The vector space dimension of $UMA(n)$ is $n^2 - n + 1$.*

Proof. We use a counting argument for the dimension. We start with the basis map matrix W that has all its ones in the first row, and then note that the basis map matrices that have all their ones in the top row, except for the first column are linearly independent. Likewise, when we add in the basis map matrices that have their ones in the first row except in the second column (excluding W since we already added it), then we still have a linearly independent set of matrices, and so on. After doing this for all the columns, we have a linearly independent set of matrices that has dimension $n(n-1)+1$ (the number of columns times the number of additional rows plus the basis map matrix that has all its ones in the first row).

To show it spans $UMA(n)$, we just note that from the set of basis map matrices above, we can form all matrices that have zeros everywhere except in one column, where there's exactly one 1, one -1, and the rest zeros (just take the two basis map matrices that have ones in the first row except at the desired column, where one has a 1 in one position and the other basis map matrix in the other position and take their difference). These matrices now allow us to take W to any other basis map matrix, and so the span of the set constructed above is $UMA(n)$. $\square$

Corollary 6.1.1. *In finite dimensions, the algebra generated by both $L(S)$ and $R(S)$ is strictly contained in $B(H)$.*

Proof. In finite dimensions, both $L(S)$ and $R(S)$ is contained in $UMA(n)$ and so the algebra they generate must be strictly contained inside $B(H)$. $\square$

Theorem 6.1.1. *For any finite dimension n, $UMA(n)$ is reflexive. Moreover, its only non trivial invariant subspace is the hyperplane*

$$M = \{(x_1, ..., x_n) \in B(H) \mid \sum_{i=1}^{n} x_i = 0\}.$$

Proof. The best way to do this is to reinterpret what it means for a matrix A to leave M invariant. Since vectors of the form $e_m - e_n$, where e_m and e_n are from the basis of H, span the subspace M, for any two basis elements e_i and e_j, the vector $A(e_i - e_j)$ must have coefficients that sum to zero, or equivalently, the sum of the entries in the e_i column of A is the same as the sum of the entries in the e_j column. Doing this for the rest of the columns, we have that the entries in each column of A must sum to the same value.

All the matrices inside $UMA(n)$ satisfies this since the basis does, so it remains to show that the matrices outside of $UMA(n)$ do not. Using the basis constructed in Proposition 6.1.1, we extend it to a basis to all by $B(H)$ by adding in the matrix units along the first row except at the last column (since we only needed to add $n-1$ of them).

Suppose we a matrix B written with respect to this vector basis and that B is invariant on M. Without loss of generality, we may assume that B is 0 on $UMA(n)$ since if B has a common sum in each column, then each column will still have a common sum after subtracting off the $UMA(n)$ part. Now we have that B is a matrix with zeros everywhere except in the first $n-1$ entries in the first row, and also that B has the same sum in each column. So the first $n-1$ columns must sum to 0 since the last column sums to zero. Thus, B is the zero matrix, and so B must be in $UMA(n)$. Thus, $UMA(n)$ must be reflexive. □

So, even though for a given finite semigroup S, $L(S)$ need not be reflexive, we see that the matrices that need to be added to make it reflexive must be linear combinations of basis map matrices.

Corollary 6.1.2. *If S is a finite, unital semigroup, and $L(S)$ is not reflexive, then $Alg(Lat(L(S))) = Alg\{L(S), B\}$, where B is a set of linear combinations of basis map matrices.*

With the above theorem complete, it seems natural to try to extend the result to infinite dimensions. Indeed, if we take the WOT closure of UMA on a separable Hilbert space we get a reflexive algebra, but it is $B(H)$.

Proposition 6.1.2. *In infinite dimensions, UMA is weakly dense.*

Proof. One reason to suspect this result is that the infinite analogue of the invariant subspace M is not a closed subspace. Then by a theorem of Nordgren, Radjavi, and Rosenthal [[4]] we know that if UMA doesn't have any invariant subspaces, then it must be $B(H)$ since it contains many finite rank operators.

However, an easier method would be to just show that UMA is weakly dense directly. For any fixed i,j, take the sequence $\{A(i,j,k)\}_{k=i+1}^{\infty}$, where $A(i,j,k)$ is the matrix that is zero everywhere except at the (i,j) entry, where it is 1, and at the (k,j) entry, where it is -1. Each $A(i,j,k)$ matrix is in UMA, and the WOT limit of the given sequence above is the matrix that is zero everywhere except at the (i,j) position, where it is 1. Thus, the WOT closure of UMA contains all the matrix units, and so, UMA is weakly dense. □

The above examination may not leave much interest left in studying UMA since in infinite dimensions it's too big. However, there is a generalization of it that is of more interest, which corresponds to attaching a sequence of weights in a certain way to the underlying Hilbert space, but these modified UMA algebras will not be discussed here.

6.2 A Multiplicative Functional for $UMA(n)$

Before we end this chapter, there is an interesting functional that arises in $UMA(n)$ for finite n. This functional is analogous to the non-trivial invariant subspace of $UMA(n)$ and will prove very useful in the next section.

Definition 6.2.1. *For any linear combination of basis map operators $X = a_0w_0 + a_1w_1 + \ldots + a_mw_m$ in $UMA(n)$, we define the basis map operator class of X as the sum of coefficients of X in any column.*

Of course, the first order of business for this new definition is to make sure it is well-defined.

Proposition 6.2.1. *The above definition of the basis map operator class is well defined (that is, all columns have the same sum), and in the above definition is equal to* $\sum_{i=0}^{m} a_i$.

Proof. To show that all columns have the same sum, we just recall that for any $X \in UMA(n)$, and standard basis vectors e_i and e_j, $X(e_i) - X(e_j) = X(e_i - e_j)$ is in the non-trivial invariant subspace. Thus the coefficient sum must be 0, so the column sum for e_i is the same as the column sum of e_j.

Finally, to show that this class is $\sum_{i=0}^{m} a_i$, we just note that all basis map matrices have exactly one 1 in each column. So for any column, the sum in that column will just be the sum of the coefficients a_i. □

The notation of basis map operator class now gives us a reasonable notation of basis map operator equivalence. Namely,

Definition 6.2.2. *The map* $\omega : UMA(n) \to \mathbf{C}$ *will be defined as the map that sends a linear combination of s to its basis map operator class. Moreover, two linear combinations of basis map operators will be called class equivalent if they have the same basis map operator class.*

Using this map, we have the follow theorem.

Theorem 6.2.1. *(Basis Map Operator Functional for* $UMA(n)$*) For any finite* n *and* $X, Y \in UMA(n)$*, the map* ω *has the following properties:*

1.) ω *induces an equivalence relation on* $UMA(n)$.

2.) $\omega(X + Y) = \omega(X) + \omega(Y)$.

3.) $\omega(XY) = \omega(X)\omega(Y)$.

4.) $\omega(cX) = c\omega(X)$.

Proof. The proof of property one follows easily from the fact that ω is well defined.

For number two, we note that X and Y are linear combinations of basis map operators, so $X+Y$ is a linear combination of basis map operators as well. Thus $\omega(X+Y)$ is just the sum of the coefficients of these basis map operators, which is the same as the coefficient sum for $\omega(X)+\omega(Y)$.

For the third property, we note that XY is also a linear combination of basis map operators. So, since a basis map operator times another basis map operator is a basis map operator, we have from the distributive property of matrix multiplication that $XY = \sum_{i,j<n} a_i b_j w_i w_j$. Thus, $\omega(XY) = \sum_{i,j<n} a_i b_j = \omega(X)\omega(Y)$.

Finally, property four comes immediately by factoring out a constant from the sum of coefficients. □

The fact that a multiplicative linear functional for $UMA(n)$ exists may be a bit surprising, but after some thought it may become clear that such a functional would be associated to the trivial character of a group. Indeed, if we restrict to the group case, then ω can be seen to be the trivial character but with an extended domain. However, because this functional would work with any semigroup (instead of group), the reasoning may not be so obvious that it should exist since there is no common one dimensional invariant subspace for all semigroup representations (unlike for groups).

The above though can be viewed in a different light - instead of viewing the one dimensional subspace, instead look at its complement. This subspace is then an invariant subspace for any semigroup representation (it is the subspace M given before as the invariant subspace of $UMA(n)$), and the above argument shows how the functional arises for it.

Before we go onto the next section and use the ω functional for the isomorphism problem, we should check to see if there are any other multiplicative functionals on $UMA(n)$. After some work, we see that there are none.

Theorem 6.2.2. *The functional ω is the only non-trivial multiplicative functional on $UMA(n)$.*

Proof. Let ϕ be a non-trivial multiplicative functional on $UMA(n)$. First, we note that the kernel of ϕ is an ideal of $UMA(n)$, and has vector dimension $n^2 - n$. We will find an explicit basis of this ideal. First we note that there is an element A in $UMA(n)$ where $\phi(A) \neq 0$, and thus, since ϕ is multiplicative, $\phi(A) = \phi(I)\phi(A)$, so $\phi(I) = 1$. Next, we let B be the matrix that is zero everywhere except in one column, where the sum in this column is zero and the diagonal entry is also zero. Since $B^2 = 0$, we must have that $\phi(B) = 0$. The vector span of all such matrices B form a $n(n-2)$ dimensional subspace of the kernel of ϕ (a basis for each of the n columns can be given by matrices with exactly two non-zero entries whose sum is zero, of which there are $n-2$ which don't include the diagonal entry.

Next, we note that for a matrix C with zeros everywhere, except for a single column (call this the k-th column), in which there is a one in the diagonal entry and a -1 somewhere else in the column, we have that $C^2 = C$. This gives us that $\phi(C)^2 = \phi(C)$, and so $\phi(C) = 0$ or 1. Assume that there is such a C that $\phi(C) = 1$. Then, for any other D which has only a single column that is non-zero, with a 1 on the diagonal entry, a -1 somewhere else, and zeros elsewhere, we take the product CD. Adding a matrix B of the form in the above paragraph, we can make any given row of D zero, meaning that we can make $CD = 0$, while not changing the value of ϕ on D (since $\phi(D + B) = \phi(D) + \phi(B) = \phi(B)$). Thus $\phi(CD) = 0$, and thus, $\phi(D)$ is zero. This means that all such matrices C must have a ϕ value of 0, except possibly one. In particular, we have now found $(n-2)n + (n-1) = n^2 - n - 1$ basis elements for the kernel of ϕ.

Now, we wish to show that $\phi(C)$ must be 0 for all C. Assume otherwise that there is a C such that $\phi(C) = 1$, and consider $C - I$. This matrix has -1's along the diagonal, except in the k-th column, where it has a -1 in some row m, and zeros everywhere else.

From here, we add on a matrix B which has zeros everywhere except in the m-th column, where it has a 1 in the diagonal entry, a -1 in the k-th row, and zeros elsewhere. The matrix $C - I + B$ is now -1 times a permutation matrix (moreover, a transposition matrix), with it's square equal to the identity. By the above paragraph, $\phi(B)$ is 0, so $\phi((C-I+B)^2) = \phi((C-I)^2) = \phi(C-I)\phi(C-I) = 0$, but $\phi((C-I+B)^2) = \phi(I)^2 = 1$, so we have a contradiction. Thus, $\phi(C)$ must be zero as well. Thus, we now have $n^2 - n$ basis elements for the kernel of ϕ, and since all of these basis vectors are also basis vectors for the kernel of ω (every column sum in these matrices is 0), and since $\phi(I) = \omega(I) = 1$, we have that $\phi(A) = \omega(A)$ for all $A \in UMA(n)$, so we are done. $\square$

Chapter 7

The Isometric Problem

In this chapter we investigate the isomorphism problem for semigroups. This can be stated as the problem of deducing when $L(S) \cong L(T)$ implies that $S \cong T$. In general for groups this need not be true, and may even depend on the field that one chooses to take the representation over, and what class of groups one considers.

In the semigroup case things are much more general, so if we hope to get any results, we will have to require some basic conditions on the given semigroups.

7.1 Basic Properties

The first such condition we should require is that the obvious map going from S to the semigroup of matrices $S_m = \{L_a | a \in S\}$ is actually an isomorphism of semigroups (after all, the idea of a representation is to embed the original object into $B(H)$). Clearly, regardless of what the semigroup looks like we will have $L_{ab} = L_a L_b$, so the map from S to $L(S)$ is a homomorphism and onto, but it does not need to be injective.

For example, consider the two element semigroup with both left multiplications acting as the identity. Under left multiplication this is seen to not be injective, hence not an isomorphism. Thus a condition should be placed on the semigroup so that for all $a, b \in S$, there exists $s \in S$ such that $as \neq bs$, which prevents left multiplication from being defined exactly the same for both elements a, b. If this condition is not met, then a and b become indistinguishable when one looks at their left multiplications, and one can think of the condition above as being similar in idea to the T_0 condition for a point set topology.

7.2 The Isomorphism Problem

Considering a semigroup with the above property, we now look to the slightly different question of whether or not two semigroups can give rise to the same algebra L under the left regular representation. If the semigroups fail the above property, then one can easily form the following (non isomorphic) semigroups,

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	d	d	d	d

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	d	d	d	d
d	d	d	d	d

the first being the semigroup that sends any word with the furthest left letter as d to d and the rest to a, while the second sends any word with the furthest left letter as c or d to d, while the rest is sent to a.

This gives a strong counter example in the sense that with the above labelings for the elements of both semigroups and that of $l^2(S)$, the left regular representation of both semigroups are indeed the same, not just isomorphic. To clarify this, we present another example with the two semigroups of order two that have identity,

.	a	b
a	a	b
b	b	a

.	$\overline{a}$	$\overline{b}$
$\overline{a}$	$\overline{a}$	$\overline{b}$
$\overline{b}$	$\overline{b}$	$\overline{b}$

While both of these semigroups map isomorphically into their left regular representations and moreover have an identity, their left regular representations are isomorphic. The transformation ϕ that sends L_a to $L_{\overline{a}}$ and L_b to $L_{\overline{a}} - 2L_{\overline{b}}$ gives an isomorphism since it is bijective, preserves the identity, and $\phi(L_b)\phi(L_b) = (L_{\overline{a}} - 2L_{\overline{b}})(L_{\overline{a}} - 2L_{\overline{b}}) = L_{\overline{a}} - 2L_{\overline{b}} - 2L_{\overline{b}} + 4L_{\overline{b}} = L_{\overline{a}} = \phi(L_b^2)$. This makes for a rather slim hope for two isomorphic representations to come from the same semigroups (in fact the above isomorphism will work with any underlying ring instead of **C**), especially since in general it is not true for groups. One result that is known is that for finite and abelian (or more generally, metaabelian[1]) groups, isomorphic integral representations implies isomorphic groups. Generalizing this to semigroups however will not be possible since the example given above again gives a very strong counter example.

To deal with these difficulties, we will have to first require the injectivity of the map from semigroup to the left regular representation. As stated before, all that this requires is that each L_a differs on at least one element of the semigroup. However, there is still a possibility that two different semigroups could map to the same representation. The problem this time is that the semigroup matrices involved may not be linearly

[1]A group G is said to be metaabelian if its commutator group $[G, G]$ is abelain

independent. Indeed, when we consider the sum

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

we see that it is still possible for two semigroups to give the exact same representation. As such, we may want the additional property that the semigroup's corresponding matrices are linearly independent. The easiest way to ensure this is to require that the semigroup has an element whose right multiplication is cancellative, since then the column that corresponds to that element will make any linear combination that is not a single matrix fail to have exactly one 1 and the rest 0 in that column. In particular, this can be done by assuming the semigroup has an identity.

As shown previously though, this doesn't give very much to work with for the isomorphism problem. Even in the group case it is not true, and the extension into semigroups would just make it even more difficult. As such, it may prove better to require a stricter form of isomorphism between the left regular representations than just an isomorphism of algebras. To inspire this, when one looks at the counter example for the two unital semigroups of order two, one may notice that the isomorphism is induced from a change of basis, and that this change of basis does not preserve inner products. Indeed the only other possible isomorphism for these two semigroups will send b to $-\overline{a}+2\overline{b}$, making these two representations not isometrically isomorphic. This seems to indicate that it could be useful to consider the isomorphism problem in a Hilbert space sense and use the inner product as a means of distinguishing representations.

As such, we pose the problem:

Problem 7.2.1. *(The Isometric Isomorphism Problem) Given two unital semigroups S and T, if there is a unitary change of basis that takes $L(S)$ to $L(T)$, must S be isomorphic to T?*

We note though that the requirement of the semigroup having a right cancellative element is essential. The first counter example given in this section will work no matter what notion of isomorphism we may require, so at the very least we have to require that two non-isomorphic semigroups can not map to the same algebra. Having a right cancellative element will make sure that if two semigroups map to the same algebra, then they also map to the same, linearly independent semigroup matrices, and so both images of the semigroups into the left regular representation are are the same, hence the semigroups are not only isomorphic, but the same with respect to the labeling from the vector space.

7.3 Isometric Problem and Basis Map Operator Class

While we will not go in a lot of detail about the isomorphism or isometric problems, we will give a partial start to working with them. We wish to get an idea of what happens when dealing with an isomorphism between two $L(S)$'s, and will use the functional ω that was introduced in Chapter 6.

Lemma 7.3.1. *Suppose there is a isomorphism ϕ between $L(S)$ and $L(T)$ for some semigroups S and T of order n. For $s \in S$, we have that $\omega(\phi(L_s))$ is an ith root of unity for some $i \leq n$, or possibly 0 if s is not invertible.*

Proof. Let $\omega(L_s) = m$. Taking powers, we have that $\phi(L_{s^k}) = \phi(L_s)...\phi(L_s)$ (k times). Thus $\omega(\phi(L_{s^k})) = m^k$. If s is not invertible, then for some $l < n$, $m^n = m^l$. Thus, $m^{n-l} = 1$ or $m = 0$. If s is invertible, then for some $k \leq n$, s^k is the identity, so $m^k = 1$, so $m = 0$ is not possible.

Here we note that if s is invertible, then k is the order of s with m as a k-th root of unity, and if s is not invertible, then m is either 0, or it is a l-th root of unity, where l is the order of the cycle on the connected component of s in the graph of L_s. □

Now, using the above, we can show that some assumptions on the isomorphisms between semigroup representations can be made without a loss of generality.

Theorem 7.3.1. *Suppose that G and H are finite groups, and that $L(G)$ is isomorphic to $L(H)$ under the map ϕ. Then there is an isomorphism ψ between $L(G)$ and $L(H)$ that preserves basis map operator class. Moreover, if ϕ respects inner products, then ψ can be made to respect inner products as well.*

Proof. We start with any isomorphism ϕ from $L(G)$ to $L(H)$, and compose ϕ with the map $M : L(H) \to L(H)$, such that for $a \in H$, $M(L_a) = \omega(\phi^{-1}(L_a))L_a$. Since G and H are groups, ω is never 0, and so ϕ composed with M is still bijective. Denoting this composition ψ, for $L_x, L_y \in L(G)$, $\psi(L_xL_y) = \omega(L_xL_y)\phi(L_xL_y) = \omega(L_x)\omega(L_y)\phi(L_x)\phi(L_y) = \omega(L_x)\phi(L_x)\omega(L_y)\phi(L_y) = \psi(L_x)\psi(L_y)$, and so ψ is still an isomorphism of representations. Since $\omega(L_a) = 1$ for all $a \in H$, ψ also preserves the basis map operator class since the coefficient $\omega(\phi^{-1}(L_a))$ changes ϕ from sending something of class k to 1 to sending it from a class of k to k again. By linearity, the preservation of basis map operator class extends to all of $L(G)$.

Finally, we wish to show that if ϕ preserves inner products, then so will ψ. Since ϕ preserving inner products means that ϕ is given by a unitary change of basis, and since M preserves inner products (as a matrix on a vector space, it is a diagonal matrix with numbers of norm 1 along the diagonal, and so as an operator on the Hilbert space, is a unitary change of basis), the composition of M with ϕ also can be written as a unitary change of basis on the Hilbert space, and so ψ preserves inner products as well, and we are done. □

Appendices

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