ECE1638 Linear Systems and Control

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These notes are intended to be more-or-less self-contained for the mythical “mathematically mature engineering student.” You need to know linear algebra and linear systems at the undergraduate level. Below are some reference texts. Most are quite advanced, but you may be able to mine them for some useful nuggets.

Chapter 1

[1] P. R. Halmos, *Finite-Dimensional Vector Spaces* — the very best, beautifully written text on linear algebra


[3] J. Conway, *A Course in Functional Analysis* — a much deeper treatment, certainly much more than you need for this course, but useful as a reference

Chapter 2


Chapter 3


Chapter 4


Chapters 5, 6, and 7 are taken from

[9] B. Francis, *A Course in $H^\infty$ Control Theory*

Chapter 8


The state-space material in Chapter 8 is taken from the paper

History

To see where this course fits into the general scheme of things, here’s a brief history of linear control system synthesis for dynamical systems.


1940’s: Military applications at MIT. Wiener and optimal filtering (see Wiener’s book, Time Series). Phillips applied Wiener’s method to design $H_2$ optimal controllers with real parameters. See Nichols, James, Phillips.


1990’s: Numerical methods to improve the design procedures, LMI.
What is this course about?

Linear systems can be modeled, in the continuous-time domain, via either differential equations, especially state equations, or input-output equations. Both viewpoints have something to offer. I would advocate that certain system concepts, such as time-invariance, causality, and certainly input-output stability, are more transparent in the input-output model. On the other hand, computations can be readily done using state model representations and MATLAB.

This course develops linear system theory in the input-output domain. The mathematical framework is operator theory in both the time domain and frequency domain. We begin in discrete time because it’s easier.

Here’s an example development in the course to give you the flavour. Consider a discrete-time system with input \( u(k) \) and output \( y(k) \). Assume the system is linear. To talk about bounded input, bounded output (BIBO) stability we need a norm for inputs and outputs. In \( \mathbb{R}^n \), all norms are equivalent, but not in signal space, which is infinite dimensional. The input norm \( \|u\| \) can be based on energy, peak value, or something else. Once we settle on a norm for \( u \) and \( y \) (actually, they don’t have to have the same norm), then BIBO stability means \( \|y\| \) is finite whenever \( \|u\| \) is finite and also \( \|y\| \leq B\|u\| \) for some constant \( B \). This is equivalent to saying that the mapping from \( u \) to \( y \) is a bounded operator. If this operator is time-invariant, then there’s a transfer function, and BIBO stability translates to some property of the transfer function. We haven’t yet assumed that the system is finite dimensional, so the transfer function could be irrational. If it is rational, so there’s a state model, then BIBO stability is independent of the norm adopted and it can be checked in the familiar way by pole locations or eigenvalue locations.

Why might you take this course? Because you like mathematical system theory. To learn input-output system theory as opposed to state-space theory. To learn some deep results such as the parametrization of all stabilizing feedback controllers.

On the other hand you should realize that, although the course was research-oriented when it was first developed in 1988, now it represents a mature subject.
Chapter 1

Function Spaces in the Time Domain

The goal of this brief, elementary chapter is to introduce Banach and Hilbert spaces and the particular example $h_2$ of square-summable discrete-time signals.

Some motivation

We motivate the subject by some examples about system gain.

Let $a \in \mathbb{R}$ and define the mapping

$$ F : \mathbb{R} \to \mathbb{R}, \quad Fx = ax $$

Think of $F$ as a “system”: input $x$, output $y = ax$. Graph $F$. Define

$$ \|F\| = \max \{ |Fx| : |x| \leq 1 \} $$

It’s clear that $\|F\| = |a|$.

Again let $a \in \mathbb{R}$ but define the “system”

$$ F : \mathbb{R}^n \to \mathbb{R}^n, \quad Fx = ax $$

Take the Euclidean norm on $\mathbb{R}^n$. Then define

$$ \|F\| = \max \{ \|Fx\| : \|x\| \leq 1 \} $$

What is $\|F\|$ in terms of $a$?

What about this harder problem: Let $A \in \mathbb{R}^{n \times n}$ and define the “system”

$$ F : \mathbb{R}^n \to \mathbb{R}^n, \quad Fx = Ax $$

Take the Euclidean norm on $\mathbb{R}^n$. Then define the norm of $F$:

$$ \|F\| = \max \{ \|Fx\| : \|x\| \leq 1 \} $$

What is $\|F\|$ in terms of $A$?
What about this even harder problem: Let
\[ G(s) = \frac{10s - 1}{s^2 + s + 10} \]
and define the system \( F \) that maps an input \( u(t) \) to an output \( y(t) \) where \( \dot{y}(s) = G(s)\dot{u}(s) \). Take the norm on \( \|u\| = \sup_{t} |u(t)| \). Then define the norm of \( F \):
\[ \|F\| = \max\{\|Fu\| : \|u\| \leq 1\} \]
What is \( \|F\| \) in terms of \( G(s) \)?

These kinds of questions motivate us to study function spaces, norms, operators, etc.

1.1 Vector space

You are assumed to be familiar with the notion of a vector space over a field, for us either the real field \( \mathbb{R} \) or the complex field \( \mathbb{C} \). Familiar examples of vector spaces are
\[ \mathbb{R}^n, \mathbb{C}^n, \mathbb{R}^{n\times m}, \mathbb{C}^{n\times m} \]
Perhaps a less familiar example is \( s(\mathbb{C}^n) \), the space of sequences written like this:
\[ x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \end{bmatrix}, \quad x_i \in \mathbb{C}^n \]
It’s convenient to write \( x \) as an infinite column vector, as shown, so that later we can multiply it by a matrix. Sometimes we’ll write \( x = (x_k) \). You can think of \( x \) as a multivariate discrete-time signal, with \( k \) as the time variable. If \( \mathbb{C}^n \) is understood or irrelevant, we may abbreviate \( s(\mathbb{C}^n) \) by \( s \).

Let \( \mathcal{X} \) be a vector space. A **subspace** \( \mathcal{V} \) of \( \mathcal{X} \) is a subset that is closed under addition and scalar multiplication. Thus \( \mathcal{V} \) is a vector space too. For example,
\[ \{x \in s : x_0 = 0\} \]
is a subspace of \( s \)—the space of signals that start at 0.

For two subspaces \( \mathcal{V} \) and \( \mathcal{W} \) define
\[ \mathcal{V} + \mathcal{W} := \{v + w : v \in \mathcal{V}, w \in \mathcal{W}\} \]
\[ \mathcal{V} \cap \mathcal{W} := \{x : x \in \mathcal{V}, x \in \mathcal{W}\} \]
It’s easy to check that this sum and intersection are subspaces.

Suppose \( \mathcal{X} \) and \( \mathcal{Y} \) are vector spaces. There’s a natural way to combine them to get a third vector space, called their **external direct sum**, denoted \( \mathcal{X} \oplus \mathcal{Y} \). As a set, \( \mathcal{X} \oplus \mathcal{Y} \) is the Cartesian product that consists of all ordered pairs, written
\[ \begin{bmatrix} x \\ y \end{bmatrix} \]
Addition and scalar-multiplication are performed componentwise.
1.2 Normed and inner-product spaces

Let $\mathcal{X}$ be a complex vector space. A norm on $\mathcal{X}$ is a function $x \mapsto \|x\|$ from $\mathcal{X}$ to $\mathbb{R}$ having the four properties:

1. $\|x\| \geq 0$
2. $\|x\| = 0$ iff $x = 0$
3. $\|cx\| = |c|\|x\|$, $c \in \mathbb{C}$
4. $\|x + y\| \leq \|x\| + \|y\|$

Then the pair $(\mathcal{X}, \|\|)$, or just $\mathcal{X}$, is called a (complex) normed space. Similarly for real normed space.

An inner-product $\langle , \rangle$ on $\mathcal{X}$ is a function from $\mathcal{X} \times \mathcal{X}$ to $\mathbb{C}$ having the four properties:

1. $\langle x, x \rangle$ is real and $\geq 0$
2. $\langle x, x \rangle = 0$ iff $x = 0$
3. for each $x$ the function $y \mapsto \langle x, y \rangle$ from $\mathcal{X}$ to $\mathbb{C}$ is linear
4. $\overline{\langle y, x \rangle} = \langle x, y \rangle$

Then the pair $(\mathcal{X}, \langle , \rangle)$, or just $\mathcal{X}$, is called a (complex) inner-product space. Similarly for real inner-product space. Such an inner-product induces a norm, namely $\|x\| := \langle x, x \rangle^{1/2}$.

Examples of inner-product spaces are:

- the space $\mathbb{C}^n$ with $\langle x, y \rangle := x^* y := \sum x_i y_i$
- the space $\mathbb{C}^{n \times m}$ with $\langle A, B \rangle := \text{trace} A^* B$

If $\mathcal{X}$ and $\mathcal{Y}$ are two inner-product spaces, their external direct sum has a natural inner-product, namely

$$\left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle := \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle$$

Two vectors $x$ and $y$ in an inner-product space are orthogonal, symbolized $x \perp y$, if $\langle x, y \rangle = 0$

The following two statements hold in an inner-product space:

- (Pythagorean theorem) if $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$
- (Cauchy-Schwarz inequality) $|\langle x, y \rangle| \leq \|x\| \|y\|$

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An important example for us is the space $h_2(\mathbb{C}^n)$, or just $h_2$ when $\mathbb{C}^n$ is understood. It’s the subspace of $s(\mathbb{C}^n)$ of sequences $x = (x_k)$ that are square-summable, i.e.,

$$\sum_{k=0}^{\infty} \|x_k\|^2 < \infty$$

Here $\|x_k\|$ is the $\mathbb{C}^n$-norm of $x_k$. The inner-product on $h_2$ is defined to be

$$\langle x, y \rangle := \sum_{0}^{\infty} x^*_ky_k$$

We should verify that the series $\sum_{0}^{\infty} x^*_ky_k$ does indeed converge when both $\sum \|x_k\|^2$ and $\sum \|y_k\|^2$ converge. It suffices to prove absolute convergence of $\sum x^*_ky_k$. (Absolute convergence in $\mathbb{C}$ implies convergence.) Set $c_k := \sum_{0}^{k} |x^*_iy_i|$. Then $c_k$ is an increasing real sequence, so it suffices to show it’s bounded. But

$$c_k = |x^*_0y_0| + \cdots + |x^*_ky_k|$$

$$\leq \|x_0\||y_0\| + \cdots + \|x_k\||y_k\|$$

by the Cauchy-Schwarz inequality in $\mathbb{C}^n$

$$\leq (\sum_{0}^{k} \|x_i\|^2)^{1/2}(\sum_{0}^{k} \|y_i\|^2)^{1/2}$$

by the Cauchy-Schwarz inequality in $\mathbb{R}^{k+1}$

$$\leq (\sum_{0}^{\infty} \|x_i\|^2)^{1/2}(\sum_{0}^{\infty} \|y_i\|^2)^{1/2}$$

The last quantity is a bound for $c_k$.

The norm on $h_2$ is denoted $\|x\|_2$, i.e.,

$$\|x\|_2 := (\sum_{0}^{\infty} \|x_k\|^2)^{1/2}$$

Now let $\mathcal{X}$ be a general normed space. A sequence $x_k$ in $\mathcal{X}$ **converges to** $x$ in $\mathcal{X}$ if the sequence $\|x - x_k\|$ of real numbers converges to zero. In this case $x_k$ is **convergent** and $x$ is its **limit**. A subspace $\mathcal{V}$ of $\mathcal{X}$ is **closed** if every sequence in $\mathcal{V}$ that converges in $\mathcal{X}$ actually has its limit in $\mathcal{V}$, i.e., $\mathcal{V}$ contains the limits of all its convergent sequences.

For example, every subspace of $\mathbb{C}^n$ is closed. To see this, let $\mathcal{V}$ be a subspace and let $\{v_1, \ldots, v_m\}$ be a basis for it. We may as well suppose these vectors are orthonormal; otherwise, orthonormalize them via Gram-Schmidt. Now extend to get a basis

$$\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$$

for all of $\mathbb{C}^n$. Again, we can suppose this basis is orthonormal. Now let $\{x_k\}$ be a sequence in $\mathcal{V}$ converging to some $x$ in $\mathbb{C}^n$. We must show that $x \in \mathcal{V}$. Being in $\mathcal{V}$, each $x_k$ has the form

$$x_k = a_{1k}v_1 + \cdots + a_{mk}v_m$$
Similarly,

\[ x = a_1v_1 + \cdots + a_mv_m + \cdots + a_nv_n \]

By orthonormality

\[ \|x_k - x\|^2 = |a_{1k} - a_1|^2 + \cdots + |a_{mk} - a_m|^2 + |a_{m+1}|^2 + \cdots + |a_n|^2 \]

This converges to zero as \( k \to \infty \). Thus

\[ a_{m+1} = \cdots = a_n = 0 \]

i.e., \( x \in \mathcal{V} \).

Two subspaces \( \mathcal{V}, \mathcal{W} \) of an inner-product space \( \mathcal{X} \) are orthogonal, written \( \mathcal{V} \perp \mathcal{W} \), if \( v \perp w \) for every \( v \) in \( \mathcal{V} \) and \( w \) in \( \mathcal{W} \). Then we write their sum as \( \mathcal{V} \bigoplus \mathcal{W} \). Note that their intersection consists solely of the zero vector. The orthogonal complement of \( \mathcal{V} \) is

\[ \mathcal{V}^\perp := \{ x \in \mathcal{X} : x \perp v \ \forall \ v \in \mathcal{V} \} \]

If \( \mathcal{V} \) is a closed subspace, then \( \mathcal{X} = \mathcal{V} \bigoplus \mathcal{V}^\perp \) (proof omitted). For a simple example, take

\[ \mathcal{V} := \{ x \in h_2 : x_0 = 0 \} \]

Then

\[ \mathcal{V}^\perp = \{ x \in h_2 : x_k = 0 \ \forall \ k > 0 \} \]

### 1.3 Banach and Hilbert spaces

Let \( \mathcal{X} \) be a normed space. A sequence \( x_k \) in \( \mathcal{X} \) is a Cauchy sequence if

\[ (\forall \epsilon > 0) (\exists N) k, l > N \Rightarrow \|x_k - x_l\| < \epsilon \]

We say \( \mathcal{X} \) is complete if every Cauchy sequence in \( \mathcal{X} \) converges in \( \mathcal{X} \). A complete normed space is called a Banach space and a complete inner-product space is called a Hilbert space. Examples of Hilbert spaces are \( \mathbb{R}^n, \mathbb{C}^n, h_2 \), and the external direct sum of two Hilbert spaces.

### 1.4 Exercises

1. Let \( \mathcal{V} := \{ x \in h_2(\mathbb{C}^n) : x_0 = 0 \} \). Prove that \( \mathcal{V} \) is closed. Give an example of a subspace of \( h_2(\mathbb{C}^n) \) that isn’t closed.

2. Let \( \mathcal{X} \) be an inner-product space, \( \mathcal{V} \) a subspace, not necessarily closed. Prove that \( \mathcal{V}^\perp \) is a closed subspace.

3. Prove that every closed subspace of a Banach space is complete.
Chapter 2

Basic System Concepts

In this chapter we look at four basic concepts: linearity, causality, time-invariance, and stability.

2.1 Linearity

Let \( U \) and \( Y \) be vector spaces. A function \( F \) from \( U \) to \( Y \) is a linear transformation if

\[
F(a_1 u_1 + a_2 u_2) = a_1 F u_1 + a_2 F u_2, \quad a_i \in \mathbb{C}, u_i \in U
\]

Recall that every linear transformation \( F : \mathbb{C}^n \to \mathbb{C}^m \) has a matrix representation \([F]\) with respect to the standard basis: Let \( \{e_i\} \) denote the standard basis for \( \mathbb{C}^n \); define \( f_j := Fe_j \), a vector in \( \mathbb{C}^m \). Then the \( j^{th} \) column of \([F]\) is by definition \( f_j \). It is then routine to prove that for every \( x \) in \( \mathbb{C}^n \)

\[
FX = [F]x
\]

The left-hand side is the linear transformation \( F \) applied to the vector \( x \), while the right-hand side is the matrix \([F]\) multiplying the \( n \)-tuple \( x \).

The formal definition of a linear system is this: a complex linear system is a triple \((F, U, Y)\), where \( U \), the input space, is \( s(\mathbb{C}^m) \), \( Y \), the output space, is \( s(\mathbb{C}^n) \), and \( F : U \to Y \) is a linear transformation. Real linear system is defined with the obvious change. The picture going along with this definition is

\[ u_k \quad F \quad y_k \]
Thus $u_k$ and $y_k$ are the input and output at time $k$. They are vector-valued, i.e., the system is multi-input and multi-output. For the input sequence $u$ the output sequence is defined to be $y := Fu$. We shall usually refer to $F$ alone as a linear system.

Let’s formally define matrix representation for a linear system $F$. We shall do it for the single-input, single-output case, i.e., $m = p = 1$, the generalization being routine. The standard basis vectors for $s$ are defined to be $e_0, e_1, \ldots$: the $i^{th}$ component of $e_i$ is 1, all the others are 0. Then $[F]$ is defined by saying that its $j^{th}$ column is $Fe_j$.

**Example 1** Moving average model.

Let the input-output relation be the equation

$$y_k = \sum_{l=0}^{\infty} F_{kl} u_l, \quad k \geq 0$$

(2.1)

where $F_{kl} \in \mathbb{C}^{p \times m}$. To obviate the need for a discussion of convergence, it is assumed that for each $k$ only finitely many of $F_{k0}, F_{k1}, \ldots$ are nonzero; that is, the sum in (2.1) has only finitely many nonzero terms. Thus in (2.1) the output at each time is a finite linear combination of the inputs at all times. Evidently, the matrix $[F]$ in this case is

$$\begin{bmatrix}
  F_{00} & F_{01} & \cdots \\
  F_{10} & F_{11} & \cdots \\
  \vdots & \vdots & \ddots
\end{bmatrix}$$

An example is the (backward) shift $S$: the defining equations are

$$y_0 = 0$$

$$y_k = u_{k-1}, \quad k \geq 1$$

Thus $S$ acts as a delay of one time unit. The matrix $[S]$ has identity blocks all along the first sub-diagonal:

$$\begin{bmatrix}
  0 & 0 & \cdots \\
  I & 0 & \cdots \\
  0 & I & \cdots \\
  \vdots & \vdots & \ddots
\end{bmatrix}$$

Another example is the truncation projection $P_k$, defined for $k \geq 0$ by

$$y_i = \begin{cases} 
  u_i, & i \leq k \\
  0, & i > k
\end{cases}$$

The corresponding matrix has $k + 1$ identity blocks:

$$\text{block diag} (I, \ldots, I, 0, \ldots)$$
Example 2 ARMA model.

Integration by the trapezoidal rule leads to a linear system modelled by the difference equation

\[ y_0 = 0 \]

\[ y_k = y_{k-1} + \frac{1}{2}(u_k + u_{k-1}), \quad k \geq 1 \]

Here the output is a linear combination of past outputs and the present and past inputs. The matrix is

\[
\begin{bmatrix}
0 & 0 & 0 & \cdots \\
1/2 & 1/2 & 0 & \cdots \\
1/2 & 1 & 1/2 & \cdots \\
1/2 & 1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Example 3 State-space model.

The input-to-output mapping is defined via

\[ x_{k+1} = A_k x_k + B_k u_k, \quad x_0 = 0 \]

\[ y_k = C_k x_k + D_k u_k \]

Here \( A_k, B_k, C_k, \) and \( D_k \) are complex matrices.

From the definition of \([F]\) it follows that

\[ Fx = [F]x \quad \text{(2.2)} \]

for every standard basis vector \( x \) of \( s \). Again, the left-hand side is the linear transformation \( F \) applied to \( x \), while the right-hand side is the matrix \([F]\) multiplying the vector \( x \). It’s not always true that (2.2) holds for every \( x \) in \( s \), but for the remainder of this chapter we assume that it does.

Finally, we want to be able to connect compatible linear systems in parallel and series. The sum of \((F,\mathcal{U},\mathcal{Y})\) and \((G,\mathcal{U},\mathcal{Y})\) is \((F + G,\mathcal{U},\mathcal{Y})\):

![Diagram of parallel and series connection of systems F and G]
Note that
\[ [F + G] = [F] + [G] \]

The composition of \((F, U, Y)\) and \((G, Y, W)\) is \((GF, U, W)\):

\[ \begin{array}{ccc}
   & F & \\
   & \rightarrow & \\
   & G & \\
\end{array} \]

Note that
\[ [GF] = [G][F] \]

### 2.2 Causality

The idea of causality is that the output at time \(k\) depends only on inputs up to time \(k\); in other words, if two inputs are equal up to time \(k\), then the two corresponding outputs should be equal up to time \(k\). The latter description leads to the formal definition, the linear system \(F\) is causal if

\[ (\forall k \geq 0)(\forall u, \tilde{u} \in \mathcal{U}) P_k u = P_k \tilde{u} \Rightarrow P_k F u = P_k F \tilde{u} \]

Introduce the notation

\[ A\mathcal{V} := \{ Av : v \in \mathcal{V} \} \]

for a function \(A\) and set \(\mathcal{V}\) in \(A\)'s domain.

**Theorem 1** For a linear system \(F\), the following conditions are equivalent:

(i) \(F\) is causal

(ii) \(P_k F (I - P_k) = 0\)

(iii) \([F]\) is lower (block) triangular

The linear transformation \(I - P_k\) annihilates the first \(k + 1\) components of a signal, so we can think of the subspace \((I - P_k)\mathcal{U}\) as the future input space starting at time \(k + 1\); similarly for \((I - P_k)\mathcal{Y}\). Thus (ii) says that “\(F\) leaves the future invariant.” Notice that if \(F\) is causal and \((F_{ij})\) is its matrix, then the input-output equation \(y = Fu\) is equivalent to

\[ y_k = \sum_{l=0}^{k} F_{kl} u_l \]
Proof of Theorem 1 Assume (i). To prove that (ii) holds start with the definition of causality and use linearity of $F$ and $P_k$ to get

$$\forall k \geq 0) \forall u \in U) P_k u = 0 \Rightarrow P_k F u = 0 \quad (2.3)$$

But $P_k u = 0$ iff $u \in (I - P_k) U$ and $P_k F u = 0$ iff $F u \in (I - P_k) \mathcal{Y}$. Thus (2.3) implies (ii).

That (ii) implies (i) follows by reversing this argument.

Now assume (ii). To prove that (iii) holds write the matrix of $F$ as

$$[F] = \begin{bmatrix} F_{00} & F_{01} & \cdots \\ F_{10} & F_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Condition (ii) for $k = 0$ is

$$F(I - P_0) U \subset (I - P_0) \mathcal{Y} \quad (2.4)$$

The matrix of $F(I - P_0)$ is

$$\begin{bmatrix} F_{00} & F_{01} & \cdots \\ F_{10} & F_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots \\ 0 & I & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & F_{01} & \cdots \\ 0 & F_{01} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad (2.5)$$

Now (2.4) implies that the first row in (2.5) equals zero, i.e., the upper right corner

$$[ F_{01} \ F_{02} \ \cdots ]$$

in the matrix $[F]$ equals zero. Similarly, the other conditions in (ii) imply that the other upper right corners in $[F]$ equal zero. Again, the implication (iii) $\Rightarrow$ (ii) follows by the reverse argument. ■

Examples of causal systems are $S$, $P_k$, and the state-space model.

2.3 Time-invariance

The previous section introduced the backward shift $S$. A related linear system is the forward shift, denoted $S^*$ for reasons soon to become apparent. The defining equation is

$$y_k = u_{k+1}, \ k \geq 0$$

It’s easy to verify that $[S^*]$ equals the transpose of $[S]$ (so $S^*$ is not causal), $S^* S = I$ ($S^*$ is a left inverse of $S$), and $SS^* = I - P_0$.

Let $F$ be a linear system. The idea of time-invariance is this: if an input $\{u_0, u_1, \cdots\}$ produces the output $\{y_0, y_1, \cdots\}$, then the input $\{0, u_0, u_1, \cdots\}$ produces an output of the form $\{w, y_0, y_1, \cdots\}$ for some $w$. Note that $w$ will equal zero if the system is causal. So
roughly speaking, time-invariance means that shifting the input shifts the output. The formal definition is that $F$ is time-invariant if $S^*FS = F$.

A matrix $A = (A_{ij})$, where $A_{ij}$ may be complex blocks, is Toeplitz if it’s constant along diagonals, i.e.,

$$A_{i+k,j+k} = A_{ij}$$

**Theorem 2** A linear system $F$ is time-invariant iff $[F]$ is Toeplitz.

**Proof** The result follows immediately upon noting that

$$[S^*][F][S] = [F] \iff [F] \text{ is Toeplitz}$$

When $F$ is time-invariant we can define $F_i = F_{i0}$, and then $y = Fu$ is equivalent to the convolution equation

$$y_i = \sum_{j=0}^{\infty} F_{i-j}u_j$$

Moreover

$$[F] = \begin{bmatrix}
F_0 & F_{-1} & F_{-2} & \cdots \\
F_1 & F_0 & F_{-1} & \cdots \\
F_2 & F_1 & F_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

Examples of time-invariant linear systems are $S, S^*$, and the state-space model with $A_k, B_k, C_k, D_k$ all constant matrices.

### 2.4 Stability

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and let $F : \mathcal{X} \to \mathcal{Y}$ be a linear transformation. Then $F$ is bounded if

$$(\exists c \in \mathbb{R})(\forall x \in \mathcal{X})\|Fx\| \leq c\|x\|$$

The least such constant $c$ is called the norm of $F$ and is denoted $\|F\|$. A bounded linear transformation is called an operator. Alternative expressions for the norm are as follows:

$$\|F\| = \inf\{c : (\forall x)\|Fx\| \leq c\|x\|\}$$

$$= \inf\{c : (\forall x \neq 0)\frac{\|Fx\|}{\|x\|} \leq c\}$$

$$= \sup_{x \neq 0} \frac{\|Fx\|}{\|x\|}$$
\[ = \sup_{\|x\| = 1} \|Fx\| \]
\[ = \sup_{\|x\| \leq 1} \|Fx\| \]

If the dimension of \( X \) is finite, then every linear transformation \( X \to Y \) is bounded. Let \( F : \mathbb{C}^n \to \mathbb{C}^m \) be linear and let \([F]\) denote its matrix with respect to the standard basis. The \textit{singular values} of \([F]\) are the square roots of the eigenvalues of the Hermitian matrix \([F]^*F\). It is a fact that

\[ \|F\| = \text{maximum singular value of } [F] \]

In infinite dimensions not all linear transformations are bounded.

A linear system \( F \) is \( h_2 \)-stable if \( F \) is an operator \( h_2 \to h_2 \). The motivation for this definition is as follows. For a discrete-time signal \( u = (u_k) \), we interpret \( \|u_k\|^2 \) as instantaneous power and \( \|u\|^2 \quad = \quad \sum \|u_k\|^2 \) as (total) energy. Thus \( h_2 \)-stability means there exists a constant \( c \) such that

\[ \text{(output energy)} \quad \leq \quad c \times \text{(input energy)} \]

It’s convenient to introduce adjoint operators at this point.

\textbf{Theorem 3} Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Hilbert spaces and \( F : \mathcal{X} \to \mathcal{Y} \) an operator. There exists a unique operator \( F^* : \mathcal{Y} \to \mathcal{X} \) satisfying the equation

\[ \langle Fx, y \rangle = \langle x, F^*y \rangle, \quad x \in \mathcal{X}, \ y \in \mathcal{Y} \]

The proof of this standard result in functional analysis is omitted. The operator \( F^* \) is called the \textit{adjoint} of \( F \). The adjoint of the shift \( S \) is the operator we’ve been denoting \( S^* \).

\section*{2.5 \textbf{Exercises}}

1. Write down \([F]\) for the state-space model in Example 3 in Section 1.

2. A linear system is strictly causal if the output is initially zero and the output at time \( k \) depends only on inputs up to time \( k - 1 \). Give a formal definition of strict causality, and state and prove the analogous theorem.

3. Consider the state-space model where the four matrices \( A_k, B_k, C_k, D_k \) are all periodic, of period \( N \). Then it’s not true that \( S^*FS = F \), but what is true?

4. Suppose \( F \) is a linear system. Show that \( F \) is causal and time-invariant iff it commutes with \( S \).

5. A linear system \( F \) is \textit{memoryless} if its matrix is (block) diagonal; thus the output at time \( k \) depends only on the input at time \( k \). Show that \( F \) is memoryless and time-invariant iff it commutes with both \( S \) and \( S^* \).
6. For this exercise we shall redefine some earlier notation. Consider sequences defined for all time, $-\infty < k < \infty$. Let $s$ denote the vector space of such sequences taking values in $\mathbb{C}^n$. Define the backward and forward shifts $S$ and $S^*$ on $s$ in the obvious way. Finally, let $F$ be a linear transformation on $s$. Show that the following three conditions are equivalent:

$$S^*FS = F$$

$F$ commutes with $S$

$F$ commutes with both $S$ and $S^*$

7. Let $F : \mathbb{C}^n \to \mathbb{C}^m$ be linear. Prove that

$$\|F\| = \text{maximum singular value of } [F]$$

8. Suppose $F$ is an operator from $\mathfrak{h}_2$ to $\mathfrak{h}_2$. Show that the matrix of $F^*$ is the complex-conjugate transpose of $[F]$. 

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Chapter 3

Function Spaces in the Frequency Domain

In this chapter we study spaces of complex-valued functions. The treatment is descriptive and mostly without proofs. Sections 1 and 2 treat scalar-valued functions; the matrix extensions are presented in Section 3. Section 4 summarizes the basic transform method relating the time and frequency domains.

3.1 Some Hilbert spaces

The Lebesgue space $L^2$ consists of all complex-valued functions defined and square-integrable on the unit circle; that is, a complex-valued function $f(e^{j\theta})$, defined for almost all $\theta$ in $[0, 2\pi)$, is in $L^2$ iff

$$\int_0^{2\pi} |f(e^{j\theta})|^2 \, d\theta < \infty$$

It’s a fact that $L^2$ is a Hilbert space under the inner-product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(e^{j\theta})g(e^{j\theta}) \, d\theta$$

An important example is that of real-rational functions in $L^2$. Let $\mathbb{R}[\lambda]$ and $\mathbb{R}(\lambda)$ denote, respectively, the ring of polynomials and the field of rational functions in $\lambda$ with real coefficients. The intersection of $L^2$ and $\mathbb{R}(\lambda)$ is denoted $RL^2$. Here and in the sequel, a prefix $R$ signifies “real-rational.” Concretely, $RL^2$ consists precisely of the real-rational functions having no poles on the unit circle.

The generic complex variable is taken to be $\lambda$. The Hardy space $H^2$ consists of all complex-valued functions $f(\lambda)$ defined and analytic on the open unit disc and having the uniform square-integrability property

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{j\theta})|^2 \, d\theta < \infty$$

$^1$Normally it would be $z$, but we can’t use $z$ for a reason to be stated later.
Thus, there is a uniform bound for the integral of $|f|^2$ around every circle inside the open unit disc.

An important fact about $H_2$ is the existence of radial limits. For $f$ in $H_2$ the limit

$$f(e^{i\theta}) := \lim_{r \to 1} f(re^{i\theta})$$

exists for almost all $\theta$ in $[0, 2\pi)$ and the boundary function $f(e^{i\theta})$ belongs to $L_2$. By identifying $f$ and its boundary function we can regard $H_2$ as a subspace of $L_2$; as such, it is closed.

The space $RH_2$ consists precisely of the real-rational functions that are analytic in the closed unit disc. To see this, let $f(\lambda) \in \mathbb{R}(\lambda)$; suppose for simplicity $f$ has only simple poles. Then $f$ is a linear combination of terms of the form

$$f_1(\lambda) = \lambda^m, \quad m \geq 0$$
$$f_2(\lambda) = \frac{1}{\lambda}$$
$$f_3(\lambda) = \frac{1}{1 - a\lambda}, \quad |a| < 1$$
$$f_4(\lambda) = \frac{1}{1 - a\lambda}, \quad |a| \geq 1$$

It is routine to check using the definition of $H_2$ that $f_1, f_3 \in RH_2$, while $f_2, f_4 \not\in RH_2$.

Cauchy’s formula holds for $H_2$: if $f \in H_2$ and $|a| < 1$, then

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(\lambda)}{\lambda - a} d\lambda$$

We saw above that an $H_2$-function has a boundary value in $L_2$, and this function has a modulus. Sometimes it is useful to be able to go the other way—to start with a positive function on the unit circle and make it the modulus of the boundary value of some $H_2$-function. This is possible if the positive function satisfies a certain condition. Let $\phi$ be a positive $L_2$-function such that $\log \phi$ is absolutely integrable on the unit circle. For $\lambda$ in the unit disc define

$$f(\lambda) := \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} \log \phi(e^{i\theta}) d\theta \right\}$$

Then $f \in H_2$ and $|f(e^{i\theta})| = \phi(e^{i\theta})$ for almost all $\theta$. (Actually, this construction gives $f$ a stronger property: It is outer, meaning that $fH_2$ is dense in $H_2$.)

Being a closed subspace of $L_2$, $H_2$ has an orthogonal complement, $H_2^\perp$. The properties of functions in $H_2^\perp$ can be derived from those in $H_2$ using the fact that

$$f(\lambda) \in H_2^\perp \iff \frac{1}{\lambda} f\left(\frac{1}{\lambda}\right) \in H_2$$

Suppose $\mathcal{X}$ is a Hilbert space and $\mathcal{V}$ a closed subspace. Then $\mathcal{X} = \mathcal{V}^\perp \oplus \mathcal{V}$, so for every $x$ in $\mathcal{X}$ there exist unique $u$ in $\mathcal{V}^\perp$ and $v$ in $\mathcal{V}$ such that $x = u + v$. Thus the operator $\Pi_\mathcal{V}$ from $\mathcal{X}$ to $\mathcal{X}$ taking $x$ to $v$ is well-defined. It’s called the orthogonal projection of $\mathcal{X}$ onto $\mathcal{V}$ and has these properties:
\[ \text{Im } \Pi_V = \mathcal{V} \]
\[ \text{Ker } \Pi_V = \mathcal{V}^\perp \]
\[ \Pi_V^2 = \Pi_V \]
\[ \Pi_V^* = \Pi_V \]
\[ \|\Pi_V\| = 1 \text{ (unless } \mathcal{V} = 0) \]

For convenience, from now on we shall make the abbreviations
\[ \Pi_1 := \Pi_{H_2^\perp} \]
\[ \Pi_2 := \Pi_{H_2} \]

As an example, consider the rational function
\[ f(\lambda) = \frac{\lambda^3 + \lambda}{-2\lambda^2 - 5\lambda + 3} \]
Having no poles on the unit circle, \( f \) belongs to \( \mathbb{R}L_2 \). The projections \( f_1 := \Pi_1 f \) and \( f_2 := \Pi_2 f \) are uniquely determined by the properties
\[ f = f_1 + f_2 \]
\[ f_1 \text{ is strictly proper and analytic in } |\lambda| \geq 1 \]
\[ f_2 \text{ is analytic in } |\lambda| < 1 \]

To get these functions, first write \( f \) as the sum of a polynomial and a strictly proper rational function. Then do a partial-fraction expansion of the latter. The polynomial goes into \( f_2 \). There results
\[ f_1(\lambda) = \frac{-0.0893}{\lambda - 0.5} \]
\[ f_2(\lambda) = -0.5\lambda + 1.25 - \frac{4.2857}{\lambda + 3} \]

### 3.2 Some Banach spaces

Elements of \( L_\infty \) are functions mapping the unit circle into \( \mathbb{C} \) whose moduli are essentially bounded. Thus, for a function \( f \) in \( L_\infty \) the modulus \(|f(e^{i\theta})|\) is uniformly bounded for almost all \( \theta \) from 0 to \( 2\pi \). The least such bound is the \( L_\infty \)-norm of \( f \):
\[ \|f\|_\infty := \sup_{\theta} |f(e^{i\theta})| \]

For example, \( \mathbb{R}L_\infty \) consists of the real-rational functions having no poles on the unit circle; thus \( \mathbb{R}L_\infty = \mathbb{R}L_2 \).
It is a fact that $L_\infty$ is a Banach space. Another useful fact is

$$\mathbb{R}[\lambda] \subset L_\infty \subset L_2$$

Both of this inclusions are strict. The second inclusion is a consequence of the inequality

$$\|f\|_2 \leq \|f\|_\infty, \quad f \in L_\infty$$

The Hardy space $H_\infty$ consists of complex-valued functions that are analytic and of bounded modulus on the open unit disc. Just as $H_2$-functions have radial limits in $L_2$, so functions in $H_\infty$ have radial limits and the limiting function belongs to $L_\infty$. In this way, $H_\infty$ is a subspace of $L_\infty$, in fact a closed subspace and hence a Banach space on its own. It is quite easy to check that $\mathbb{R}H_\infty = \mathbb{R}H_2$. The inclusions analogous to those above are

$$\mathbb{R}[\lambda] \subset H_\infty \subset H_2$$

and again both are strict.

### 3.3 Matrix spaces

All the frequency-domain spaces in the previous two sections consisted of complex-valued, i.e., scalar-valued, functions. When we want to express this fact we shall write $L_2(\mathbb{C})$ for $L_2$, and similarly for the others. In this section we extend these spaces to matrix-valued functions.

First the space $L_2(\mathbb{C}^{n \times m})$. Its elements are $n \times m$ complex matrix-valued functions each entry of which is a scalar-valued $L_2$-function. The inner-product on $L_2(\mathbb{C}^{n \times m})$ is

$$< F, G > := \frac{1}{2\pi} \int_0^{2\pi} \text{trace} F(e^{j\theta})^*G(e^{j\theta})d\theta$$

Here superscript $^*$ means complex-conjugate transpose. It follows that the $L_2$-matrix norm is

$$\|F\|_2 = (\sum_{i,j} \|f_{ij}\|_2^2)^{1/2}$$

The vector-valued space $L_2(\mathbb{C}^n)$ is the special case $m = 1$. The subspaces $H_2(\mathbb{C}^{n \times m})$ and $H_2(\mathbb{C}^{n \times m})^\perp$ are defined in the obvious way.

Finally, the space $L_\infty(\mathbb{C}^{n \times m})$ consists of matrices whose entries are scalar-valued $L_\infty$-functions, and the matrix norm is

$$\|F\|_\infty := \sup_\theta \|F(e^{j\theta})\|$$

Recall that the latter norm is maximum singular value. The subspace of matrices analytic in the open unit disc is $H_\infty(\mathbb{C}^{n \times m})$.

When the dimensions are irrelevant, $L_2(\mathbb{C}^{n \times m})$ etc. are written simply $L_2$ etc.
3.4 The $\lambda$-transform

Recall from Chapter 1 that $h_2$ is the space of square-summable sequences defined for non-negative times. We shall need two additional spaces. Define $h_2^\perp$ to be the space of square-summable sequences defined for negative times, $k = \ldots, -2, -1$. A signal $x$ in $h_2^\perp$ is written

$$\begin{bmatrix}
\vdots \\
x_{-2} \\
x_{-1}
\end{bmatrix}$$

and has the property

$$\sum_{-\infty}^{-1} |x_k|^2 < \infty$$

This space is a Hilbert space under the inner-product

$$< x, y > = \sum_{-\infty}^{-1} \overline{x_k}y_k$$

Also, define $l_2$ to be the external direct sum of $h_2^\perp$ and $h_2$:

$$l_2 := h_2^\perp \oplus h_2$$

Elements of $l_2$ will be written

$$\begin{bmatrix} x \\ y \end{bmatrix}, \quad x \in h_2^\perp, y \in h_2$$

With the inherited inner-product, $l_2$ is a Hilbert space, $h_2$ is a closed subspace, and $h_2^\perp$ is its orthogonal complement.

Recall that the (two-sided) $z$-transform of $x$ in $l_2$ is

$$\sum_{k=-\infty}^{\infty} x_k z^{-k}$$

For us it’s more convenient to use $\lambda := z^{-1}$. So define the $\lambda$-transform of $x$ to be

$$\hat{x}(\lambda) := \sum_{k=-\infty}^{\infty} x_k \lambda^k$$

The theorem coming up, a combination of the Riesz-Fischer theorem and Parseval’s equality, connects the time-domain Hilbert spaces to the frequency-domain Hilbert spaces. It is compactly stated via the notion of isomorphism. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. An isomorphism from $\mathcal{X}$ to $\mathcal{Y}$ is a linear transformation having the two properties
it is surjective
it preserves inner-products

Such a function automatically has the further properties
it preserves norms
it is injective
it is bounded
it has a bounded inverse

If such an isomorphism exists, then \( \mathcal{X} \) and \( \mathcal{Y} \) are isomorphic.

**Theorem 1** The \( \lambda \)-transform is an isomorphism from \( l_2 \) onto \( L_2 \); it maps \( h_2 \) onto \( H_2 \) and \( h_2^\perp \) onto \( H_2^\perp \).

### 3.5 Summary

**Hilbert spaces**

- \( L_2 \): square-integrable functions on the unit circle
- \( H_2 \): subspace of \( L_2 \) of functions analytic in the open unit disc
- \( H_2^\perp \): subspace of \( L_2 \) of functions analytic in the exterior of the closed unit disc and zero at infinity

**Banach spaces**

- \( L_\infty \): essentially bounded functions on the unit circle
- \( H_\infty \): subspace of functions analytic in the open unit disc

**real-rational spaces**

- \( RL_2 = RL_\infty \): real-rational functions with no poles on the unit circle
- \( RH_2 = RH_\infty \): real-rational functions with no poles in the closed unit disc (include the polynomials)
- \( RH_2^\perp \): strictly-proper real-rational functions with no poles on the unit circle or outside the unit disc
3.6 Exercises

1. Fix $a$ in the open unit disc. Find a function $f$ in $\mathbf{H}_2$ such that for every $g$ in $\mathbf{H}_2$

$$<f, g> = g(a)$$

2. Show that $\mathbf{RH}_2^\perp$ consists precisely of the strictly proper real-rational functions that are analytic in $|\lambda| \geq 1$.

3. Define $f(\lambda) = 2\lambda - 1$ and $\mathcal{V} = f \mathbf{H}_2$. Show that $\mathcal{V}$ is a closed subspace of $\mathbf{H}_2$. What is the dimension of $\mathcal{V}^\perp$, the orthogonal complement in $\mathbf{H}_2$? Find a basis for $\mathcal{V}^\perp$.

4. Find the projections in $\mathbf{H}_2$ and $\mathbf{H}_2^\perp$ of

$$\begin{bmatrix}
\frac{\lambda^2(\lambda+3)}{2\lambda^2-5\lambda+2} \\
\frac{1}{\lambda^2-2\lambda} \\
\end{bmatrix}$$

5. Define

$$F(\lambda) := \begin{bmatrix} \lambda & 0 \\ 0 & 2\lambda - 1 \end{bmatrix}$$

Let $\mathcal{V} = F \mathbf{H}_2$ and let $\mathcal{V}^\perp$ be its orthogonal complement in $\mathbf{H}_2$. Find a basis for $\mathcal{V}^\perp$.

6. Compute the $L_\infty$-norm of

$$\begin{bmatrix}
\frac{\lambda^2+1}{\lambda^2-\lambda-6} & 1 \\
\frac{\lambda}{\lambda^2-\lambda} & \lambda \\
\end{bmatrix}$$

7. Prove that if $F \in \mathbf{H}_\infty(\mathbb{C}^{n \times m})$, then

$$\|F\|_2 \leq \sqrt{m}\|F\|_\infty$$

so that $F \in \mathbf{H}_2(\mathbb{C}^{n \times m})$. 

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Chapter 4

Time-Invariant Systems

In this chapter we study time-invariant linear systems and their transfer functions. Transfer functions of causal and non-causal stable systems are characterized, and the problem is treated of finding the distance from a non-causal stable system to the nearest causal stable one. All the systems in this chapter are linear and time-invariant.

4.1 Transfer functions

Suppose $F$ is a time-invariant linear system with matrix

$$[F] = \begin{bmatrix} F_0 & F_{-1} & \cdots \\ F_1 & F_0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

The corresponding convolution equation is

$$y_k = \sum_{l=0}^{\infty} F_{k-l} u_l, \quad k \geq 0 \quad (4.1)$$

where

$$y_k \in \mathbb{C}^p, \quad u_k \in \mathbb{C}^m, \quad F_k \in \mathbb{C}^{p \times m}$$

The transfer function (tf) of $F$ is defined to be

$$\hat{F}(\lambda) := \sum_{-\infty}^{\infty} F_k \lambda^k$$

Associated with the tf is a region of convergence (roc): Let

$$r_2 := \text{radius of convergence of } \sum_{0}^{\infty} F_k \lambda^k = (\lim \sup \|F_k\|^{1/k})^{-1}$$
\[
\frac{1}{r_1} := \text{radius of convergence of } \sum_{k=1}^{\infty} F_k z^k
\]
i.e.,
\[
 r_1 = \lim \sup \| F_k \|^{1/k}
\]
Then the roc for \( \hat{F}(\lambda) \) is defined to be the annulus
\[
r_1 < |\lambda| < r_2
\]
Thus \( \hat{F} \) exists only if \( r_1 < r_2 \), and then \( \hat{F}(\lambda) \) is analytic in the roc. We assume from now on that \( \hat{F} \) exists. Note that if \( F \) is causal, then the roc is a disc, \( |\lambda| < r_2 \). Also, if \( F \) is strictly causal, then \( \hat{F}(0) = 0 \).
These definitions are illustrated by the following examples. For the first take
\[
F_k = \begin{cases} 
1, & k \geq -1 \\
0, & k < -1
\end{cases}
\]
Then \( r_1 = 0, r_2 = 1, \) and
\[
\hat{F}(\lambda) = \frac{1}{\lambda(1 - \lambda)}
\]
This system is not causal, and this is reflected in the fact that the roc is not a disc.
The second example is
\[
F_k = \begin{cases} 
0, & k \geq 0 \\
2^k, & k < 0
\end{cases}
\]
Here \( r_1 = 1/2, r_2 = \infty, \) and
\[
\hat{F}(\lambda) = \frac{1}{2\lambda - 1}
\]
Again, the system is not causal.
The third example is the state-space model
\[
x_{k+1} = Ax_k + Bu_k \\
y_k = Cx_k + Du_k
\]
The tf is
\[
\hat{F}(\lambda) = D + \lambda CB + \lambda^2 CAB + \cdots \\
= D + \lambda C(I + \lambda A + \cdots)B \\
= D + \lambda C(I - \lambda A)^{-1}B
\]
The latter rational matrix is denoted
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]
To get the roc of the series \( I + \lambda A + \cdots \) we invoke Gelfand’s formula:
\[
\lim_{k \to \infty} \| A^k \|^{1/k} = \rho(A) := \text{spectral radius of } A
\]
Thus the roc is the disc \( |\lambda| < 1/\rho(A) \).
We continue with the system of the previous section, but with the following assumptions:

(i) $F$ is causal and $\hat{F} \in H_2$

(ii) the input $u$ is standard white noise

The second assumption means that $u_k$ is a random vector with zero mean and unity covariance matrix and that $u_k$ and $u_l$ are uncorrelated for $k \neq l$. Under these assumptions the output signal $y$ has the following properties.

**Theorem 1** $y$ is (wide-sense) stationary, it has zero mean, and its root-mean-square value equals $\|\hat{F}\|_2$.

**Proof** The first two properties are routine to prove. For the third, we first compute the covariance matrix. Let $E$ denote the expectation operator. Then

$$Ey_k y_k^* = E \left( \sum_i F_{k-i} u_i \right) \left( \sum_j F_{k-j} u_j \right)^*$$

$$= E \sum_i \sum_j F_{k-i} u_i u_j^* F_{k-j}^*$$

$$= \sum_i F_{k-i} F_{k-i}^*$$

$$= \sum_i F_i F_i^*$$

Thus the mean-square value of $y$ is

$$Ey_k y_k^* = \text{trace} \left( Ey_k y_k^* \right)$$

$$= \text{trace} \sum_i F_i F_i^*$$

$$= \sum_i \text{trace} F_i^* F_i$$

The last expression equals the square of the $h_2$-norm of the sequence $F_0, F_1, \ldots$. But this equals $\|\hat{F}\|_2^2$ by Parseval’s equality. ■

There is an analogous deterministic result in the single-input, single-output case. Suppose $u$ is the unit pulse, i.e., $\hat{u}(\lambda) = 1$, and, as above, $\hat{F} \in H_2$. Then $y \in h_2$ and $\|y\|_2 = \|\hat{F}\|_2$. 23
In words: the $H_2$-norm of the tf equals the square-root of the energy of the output when the input is the unit pulse.

We complete this section with two ways to compute the $H_2$-norm of a rational tf. First is the familiar **method of residues**. Suppose $\hat{F}$ is a scalar-valued function in $RH_2$. Then

$$\|\hat{F}\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |\hat{F}(e^{j\theta})|^2 d\theta = \frac{1}{2\pi j} \oint \frac{1}{\lambda} \hat{F}(\lambda) \hat{F} \left( \frac{1}{\lambda} \right) d\lambda$$

The latter expression equals the sum of the residues of

$$\frac{1}{\lambda} \hat{F}(\lambda) \hat{F} \left( \frac{1}{\lambda} \right)$$

at its poles in the open unit disc.

Second is the **state-space method**. Suppose

$$\hat{F}(\lambda) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $\rho(A) < 1$. Define the controllability grammian

$$L_c := \sum_{i=0}^{\infty} A^i BB' A'^i$$

Then

$$\|\hat{F}\|_2 = [\text{trace} \ (DD' + C L_c C')]^{1/2}$$

The proof of this is left as an exercise.

### 4.3 tf's for causal systems

Consider a system $F$ with transfer function $\hat{F}$. If $F$ is causal, then its matrix is lower triangular and, consequently, the roc of $\hat{F}$ is some disc. It turns out that if $F$ is $h_2$-stable, then the closure of the roc contains the open unit disc and $\hat{F} \in H_\infty$.

**Theorem 2** The function $F \mapsto \hat{F}$ maps the space of time-invariant causal $h_2$-stable systems into $H_\infty$. This function is linear, bijective, and norm-preserving.

**Proof** Some parts of this theorem are fairly hard to prove. To avoid getting bogged down in technicalities, we’ll take some shortcuts.

To see that the co-domain of the mapping $F \mapsto \hat{F}$ is $H_\infty$, we’ll simplify by assuming that the tf is real-rational and scalar-valued. Apply the unit pulse as input. Then the output equals the first column of $[F]$ and belongs to $h_2$. The $\lambda$-transform of this output equals $\hat{F}(\lambda)$, which therefore belongs to $RH_2$. So $\hat{F} \in RH_2 = RH_\infty$.

Linearity and injectivity are easy.
To prove surjectivity, start with \( \hat{F} \) in \( H_\infty \). Thus \( \hat{F} \) has a power series at \( \lambda = 0 \):

\[
\hat{F}(\lambda) = F_0 + \lambda F_1 + \cdots
\]

It is claimed that \( \hat{F}H_2 \subset H_2 \). To prove this, let \( \hat{u} \in H_2 \) and define \( \hat{y} := \hat{F} \hat{u} \). Since \( \hat{u} \) and \( \hat{F} \) are analytic in the open unit disc, so is \( \hat{y} \). Moreover,

\[
\|\hat{y}\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \|\hat{F}(e^{j\theta}) \hat{u}(e^{j\theta})\|^2 d\theta
\]

\[
\leq \|\hat{F}\|_\infty^2 \|\hat{u}\|_2^2
\]

Thus \( \hat{y} \in L_2 \). Being analytic and square-integrable, \( \hat{y} \in H_2 \). This proves the claim.

Define the linear transformation \( F \) on \( s \)

\[
\begin{bmatrix}
  u_0 \\
  u_1 \\
  \vdots
\end{bmatrix} \mapsto \begin{bmatrix}
  F_0 & 0 & \cdots \\
  F_1 & F_0 & \cdots \\
  \vdots & \vdots & \ddots
\end{bmatrix} \begin{bmatrix}
  u_0 \\
  u_1 \\
  \vdots
\end{bmatrix}
\]

In view of the claim, \( F \) maps \( h_2 \) to \( h_2 \). Moreover, from (4.2) \( \|F\| \leq \|\hat{F}\|_\infty \), so \( F \) is bounded, i.e., \( h_2 \)-stable. This proves surjectivity.

Finally, to prove that the function is norm-preserving, it remains to show that \( \|F\| \geq \|\hat{F}\|_\infty \). Here again we’ll take a shortcut by assuming that \( F \) is scalar-valued and continuous on the unit circle.

Let \( \epsilon \) and \( \delta \) be positive numbers. Choose an interval \( (\theta_1, \theta_2) \) such that

\[
\theta \in (\theta_1, \theta_2) \Rightarrow |\hat{F}(e^{j\theta})| \geq \|\hat{F}\|_\infty - \epsilon
\]

Then let \( \hat{u} \) be a function in \( H_2 \) having the boundary magnitude

\[
|\hat{u}(e^{j\theta})| = \begin{cases}
  c, & \theta \in (\theta_1, \theta_2) \\
  \delta, & \text{else}
\end{cases}
\]

(Such a function exists as discussed in Section 3.1.) Here \( c \) is chosen so that \( \|\hat{u}\|_2 = 1 \), i.e.,

\[
\frac{1}{2\pi} \left\{ c^2(\theta_2 - \theta_1) + \delta^2[2\pi - (\theta_2 - \theta_1)] \right\} = 1
\]

Then

\[
\|\hat{F}\hat{u}\|_2^2 \geq \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} |\hat{F}(e^{j\theta})\hat{u}(e^{j\theta})|^2 d\theta
\]

\[
\geq \frac{1}{2\pi} c^2(\theta_2 - \theta_1)(\|\hat{F}\|_\infty - \epsilon)^2 \quad \text{from (4.3) and (4.4)}
\]

\[
= \left\{ 1 - \delta^2 \frac{2\pi - (\theta_2 - \theta_1)}{2\pi} \right\} (\|\hat{F}\|_\infty - \epsilon)^2 \quad \text{from (4.5)}
\]

\[
25
\]
Since the last expression is independent of \( \hat{u} \), we get
\[
\|F\|^2 \geq \left\{ 1 - \delta^2 \frac{2\pi - (\theta_2 - \theta_1)}{2\pi} \right\} (\|\hat{F}\|_{\infty} - \epsilon)^2
\]
But \( \delta \) was arbitrary, so
\[
\|F\| \geq \|\hat{F}\|_{\infty} - \epsilon
\]
Since \( \epsilon \) was arbitrary too, \( \|F\| \geq \|\hat{F}\|_{\infty} \). \( \blacksquare \)

Some points raised in the above proof are worth emphasizing. Let \( F \) be a causal \( \mathbf{h}_2 \)-stable system and \( \hat{F} \) its tf. The input and output are related by either of the equivalent equations

- time-domain: \( y = Fu \)
- frequency-domain: \( \hat{y} = \hat{F}\hat{u} \)

In other words, \( \hat{F}u = \hat{F}\hat{u} \). This is equivalent to commutativity of the diagram

\[
\begin{array}{ccc}
\mathbf{h}_2 & \xrightarrow{F} & \mathbf{h}_2 \\
\downarrow & & \downarrow \\
\mathbf{H}_2 & \xrightarrow{M_{\hat{F}}} & \mathbf{H}_2
\end{array}
\]

The two vertical arrows stand for \( \lambda \)-transformation, and \( M_{\hat{F}} \) is the operator of multiplication by \( \hat{F} \):
\[
M_{\hat{F}}g := \hat{F}g
\]
Since \( \lambda \)-transformation preserves norms, \( \|F\| = \|M_{\hat{F}}\| \). Thus from the theorem we have
\[
\|\hat{F}\|_{\infty} = \|F\| = \|M_{\hat{F}}\|
\]

We conclude this section with a remark about causal but unstable systems. In general the roc for \( \hat{F} \) is a disc, \( |\lambda| < r_2 \). If \( r_2 < 1 \), then \( \hat{F} \notin \mathbf{H}_\infty \), so by the theorem \( F \) is not \( \mathbf{h}_2 \)-stable. For example, if \( \hat{F} \) is rational, then \( F \) is \( \mathbf{h}_2 \)-stable iff \( \hat{F} \) has no poles in the closed unit disc. It is emphasized that this assumes \( F \) is causal.
4.4 tfs for non-causal systems

If $F$ is non-causal, then its matrix is not lower triangular and the roc of $\hat{F}$ is only an annulus, not a disc. However if $F$ is $h_2$-stable, then the closure of this annulus contains the unit circle.

**Theorem 3** The function $F \rightarrow \hat{F}$ maps the space of time-invariant $h_2$-stable systems into $L_\infty$. This function is linear, bijective, and norm-preserving.

The proof is similar to the one in the previous section.

Let $F$ be $h_2$-stable. Its matrix looks like

$$[F] = \begin{bmatrix} F_0 & F_{-1} & \cdots \\ F_1 & F_0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

and the corresponding convolution equation is

$$y_k = \sum_{l=0}^{\infty} F_{k-l}u_l, \quad k \geq 0$$

Extend this equation backwards in time and allow $u$ to belong to $l_2 = h_2^\perp \oplus h_2$ instead of just $h_2$. We get

$$y_k = \sum_{l=-\infty}^{\infty} F_{k-l}u_l, \quad -\infty < k < \infty$$

This equation corresponds to an extended operator $F_e$ with matrix

$$[F_e] = \begin{bmatrix} \vdots & \vdots & \cdots \\ \cdots & F_{-1} & F_{-2} & \cdots \\ \cdots & F_0 & F_{-1} & \cdots \\ \cdots & F_1 & F_0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

This operator, an extension of $F$, maps $l_2$ to itself. It is easily checked that the two equations

- time-domain $y = F_eu$
- frequency-domain $\hat{y} = \hat{F}\hat{u}$

are equivalent, i.e., $\hat{F}_e u = \hat{F}\hat{u}$. The corresponding commutative diagram is
Again, the two vertical arrows stand for \( \lambda \)-transformation. We have the following equalities:

\[
\| \hat{F} \|_{\infty} = \| F \| = \| F_e \| = \| \hat{M} F \|
\]

The first equality is from the theorem, the third from the commutative diagram, and the second is left as an exercise.

This section concludes with a caution. If the tf is given and it belongs to \( L_{\infty} \) but not \( H_{\infty} \), does it represent an unstable causal system or a stable non-causal system? The answer is that we don’t know. We must also be given the roc to make this decision. Consider for example the tf

\[
\hat{F}(\lambda) = \frac{1}{2\lambda - 1}
\]

having a pole in the unit disc at \( \lambda = 1/2 \). If the roc is the disc \( |\lambda| < 1/2 \), then \( F \) is causal but not \( h_2 \)-stable; if the roc is the (semi-infinite) annulus \( |\lambda| > 1/2 \), then \( F \) is \( h_2 \)-stable but not causal.

### 4.5 Nehari’s theorem

Consider the two questions: What is the distance from a given stable non-causal system to the nearest stable causal one? What is the distance from a given unstable causal system to the nearest stable causal one? Suitably posed, these two questions are equivalent. This section presents time- and frequency-domain versions of a solution.

#### 4.5.1 Time-domain

Suppose \( F \) is \( h_2 \)-stable. As in the previous section, bring in its matrix

\[
[F] = \begin{bmatrix}
F_0 & F_{-1} & \cdots \\
F_1 & F_0 & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\]

(4.6)
and its extension $F_e$. The *Hankel operator* corresponding to $F$ is

\[ \Gamma_F : h_2 \to h_2^\perp \]

\[ \Gamma_F u := \Pi_{h_2^\perp} F_e u \]

The matrix of this operator is

\[ [\Gamma_F] = \begin{bmatrix} \vdots & \vdots & \cdots \\ F_{-2} & F_{-3} & \cdots \\ F_{-1} & F_{-2} & \cdots \end{bmatrix} \quad (4.7) \]

Comparison of (4.6) and (4.7) shows that $[\Gamma_F]$ is the limit of the upper right-hand corners of $[F]$. It’s instructive to look at the convolution equations corresponding to the three operators $F$, $F_e$, and $\Gamma_F$:

- $F : y_k = \sum_{\ell=0}^{\infty} F_{k-\ell} u_\ell, \quad k \geq 0$
- $F_e : y_k = \sum_{\ell=\infty}^{\infty} F_{k-\ell} u_\ell, \quad -\infty < k < \infty$
- $\Gamma_F : y_k = \sum_{\ell=0}^{\infty} F_{k-\ell} u_\ell, \quad k < 0$

The last equation gives the output before time 0 in terms of the input at and after time 0. In this way we can think of the Hankel operator as mapping present and future inputs to past outputs. Consequently, if $F$ is causal, then $\Gamma_F = 0$.

If $F$ and $G$ are $h_2$-stable systems, the *distance* between them is $\| F - G \|$.

**Theorem 4** *The distance from an $h_2$-stable system $F$ to the nearest causal $h_2$-stable system equals $\| \Gamma_F \|$. Moreover, the distance is attained.*

For an outline of a proof, we need some notation. Suppose $A$ and $B$ are operators on Hilbert spaces, having the same co-domain, say

- $A : \mathcal{X} \to \mathcal{Z}$, \quad $B : \mathcal{Y} \to \mathcal{Z}$

Then there’s a natural operator mapping the external direct sum $\mathcal{X} \oplus \mathcal{Y}$ to $\mathcal{Z}$, namely

\[ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto Ax + By \]

This operator is denoted $[ A \ B ]$. There’s a similar construction for two operators having the same domain.

**Lemma 1** *Suppose $A$, $B$, and $C$ are operators on Hilbert space. Then*

\[ \min_X \| [ A \ B ] \| = \max \left\{ \| [ A \ B ] \| , \| [ B \ C ] \| \right\} \]
Here the minimum is over all Hilbert space operators $X$.

A general proof of this lemma would require too many extra tools. To give some comprehension of the lemma, below is a simple proof in the very special case that the Hilbert space equals $\mathbb{R}$, the reals. So let $a$, $b$, $c$ be real numbers. We want to show that

$$
\min_x \left\| \begin{bmatrix} a & b \\ x & c \end{bmatrix} \right\| = \max \left\{ \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| , \left\| \begin{bmatrix} b \\ c \end{bmatrix} \right\| \right\}
$$

where the minimum is over all real numbers $x$. Thus three elements of a $2 \times 2$ real matrix are fixed, the fourth is to be chosen to minimize the norm, and the minimum equals the maximum norm of the fixed row and the fixed column. Assume without loss of generality that

$$
\left\| \begin{bmatrix} b \\ c \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|
$$

i.e., $|c| \geq |a|$. Since

$$
\left\| \begin{bmatrix} a & b \\ x & c \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} b \\ c \end{bmatrix} \right\|
$$

for every $x$, it suffices to construct an $x$ to achieve equality. To do this, simply choose $x$ so that

$$
\begin{bmatrix} a \\ x \end{bmatrix} \perp \begin{bmatrix} b \\ c \end{bmatrix}
$$

Looking at these two vectors in the plane, i.e.,

we see that (because $|c| \geq |a|$)

$$
\left\| \begin{bmatrix} a \\ x \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} b \\ c \end{bmatrix} \right\|
$$

(4.9)
Now we have
\[
\left\| \begin{bmatrix} a & b \\ x & c \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} a & x \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ x & c \end{bmatrix} \right\|
\]
\[
= \left\| \begin{bmatrix} a^2 + x^2 & 0 \\ 0 & b^2 + c^2 \end{bmatrix} \right\| \quad \text{by (4.8)}
\]
\[
= b^2 + c^2 \quad \text{by (4.9)}
\]
\[
= \left\| \begin{bmatrix} b \\ c \end{bmatrix} \right\|^2
\]

**Proof of theorem** From Exercise 5, \(F\) and \(F_e\) have equal norm. Hence \(F\) and \(F_e|_{h_2}\), the restriction of \(F_e\) to \(h_2\), have equal norm. The matrix of the latter operator is
\[
[F_e|_{h_2}] = \begin{bmatrix}
\vdots & \vdots \\
F_{-1} & F_{-2} & \cdots \\
F_0 & F_{-1} & \cdots \\
F_1 & F_0 & \cdots \\
\vdots & \vdots 
\end{bmatrix}
\]
So it suffices to prove that the minimum of \(\|(F_e - G_e)|_{h_2}\|\) over all causal \(h_2\)-stable systems \(G\) equals \(\|\Gamma_F\|\). For such \(G\) we have
\[
[(F_e - G_e)|_{h_2}] = \begin{bmatrix}
\vdots & \vdots \\
F_{-1} & F_{-2} & \cdots \\
F_0 - G_0 & F_{-1} & \cdots \\
F_1 - G_1 & F_0 - G_0 & \cdots \\
\vdots & \vdots 
\end{bmatrix}
\]
The upper part of this matrix is \([\Gamma_F]\). Hence
\[
\|(F_e - G_e)|_{h_2}\| \geq \|\Gamma_F\|
\]
To achieve equality, construct \(G\) by specifying first \(G_0\), then \(G_1\), etc. The idea is to move down the above matrix, \([(F_e - G_e)|_{h_2}]\), row by row without increasing the norm. First, choose \(G_0\) such that the norm of the operator with matrix
\[
\begin{bmatrix}
\vdots & \vdots \\
F_{-1} & F_{-2} & \cdots \\
F_0 - G_0 & F_{-1} & \cdots 
\end{bmatrix}
\]
equals \( \| \Gamma_F \| \); this is possible by virtue of the lemma. Next choose \( G_1 \) such that the norm of the operator with matrix
\[
\begin{bmatrix}
\vdots & \vdots \\
F_{-1} & F_{-2} & \cdots \\
F_0 - G_0 & F_{-1} & \cdots \\
F_1 - G_1 & F_0 - G_0 & \cdots \\
\end{bmatrix}
\]
equals \( \| \Gamma_F \| \); this is possible for the same reason. Continue \textit{ad infinitum}. ■

4.5.2 Frequency domain

Let \( \hat{F} \in L_\infty \). Then \( \hat{F} H_2 \subset L_2 \) (cf. the proof of the theorem in Section 4.3). The \textit{Hankel operator} corresponding to \( \hat{F} \) is
\[
\Gamma_{\hat{F}} : H_2 \to H_2^+
\]
\[
\Gamma_{\hat{F}} g := \Pi_1 \hat{F} g
\]
(Recall that \( \Pi_1 \) is the orthogonal projection from \( L_2 \) onto \( H_2^+ \).) Sometimes \( \hat{F} \) is called the \textit{symbol} of its Hankel operator. Note that if \( \hat{F} \in H_{\infty} \), then \( \Gamma_{\hat{F}} = 0 \). The relationship between the two Hankel operators is exhibited in this commutative diagram:

\[
\begin{array}{ccc}
H_2^+ & \xrightarrow{\Gamma_{\hat{F}}} & H_2 \\
\downarrow_{\Gamma_{\hat{F}}} & & \downarrow_{\Gamma_{\hat{F}}} \\
H_2^+ & \xrightarrow{\Gamma_F} & H_2
\end{array}
\]

The \textit{distance} between two functions in \( L_\infty \) is the \( L_\infty \)-norm of their difference.

\textbf{Theorem 5} The distance from a function \( \hat{F} \) in \( L_\infty \) to the nearest function in \( H_{\infty} \) equals \( \| \Gamma_{\hat{F}} \| \) and the distance is attained.

4.6 State-space computation of Hankel-norm

The subject of this section is how to compute numerically the norm of a Hankel operator with real-rational symbol. So let \( \hat{F} \in RL_\infty \). The \( H_2^+ \)-component of \( \hat{F} \) has the form
\[
\Pi_1 \hat{F}(\lambda) = \cdots + \frac{1}{\lambda^2} F_{-2} + \frac{1}{\lambda} F_{-1}
\]
and then
\[
[\Gamma_F] = \begin{bmatrix}
\vdots & \vdots \\
F_{-2} & F_{-3} & \cdots \\
F_{-1} & F_{-2} & \cdots \\
\end{bmatrix}
\]
Being real-rational, \(\Pi_1 \hat{F}\) has a minimal realization:
\[
\Pi_1 \hat{F}(\lambda) = C(\lambda - A)^{-1}B
\]
Let \(n\) denote the dimension of \(A\) and observe that \(\rho(A) < 1\) because \(\Pi_1 \hat{F}\) is analytic in the exterior of the unit disc. Then we have
\[
[\Gamma_F] = \begin{bmatrix}
\vdots & \vdots & \cdots \\
CAB & CA^2B & \cdots \\
CB & CAB & \cdots \\
\vdots & \vdots \\
CA & \cdots \\
C & \cdots \\
\end{bmatrix} = \begin{bmatrix}
\vdots \\
B & AB & \cdots \\
\end{bmatrix}
\]
This equation leads us to define two auxiliary operators: the controllability operator
\[
\Psi_c : \mathbb{h}_2 \to \mathbb{C}^n, \quad [\Psi_c] = \begin{bmatrix}
B & AB & \cdots \\
\end{bmatrix}
\]
and the observability operator
\[
\Psi_o : \mathbb{C}^n \to \mathbb{h}_2^\perp, \quad [\Psi_o] = \begin{bmatrix}
\vdots \\
CA \\
C \\
\end{bmatrix}
\]
Clearly
\[
\Gamma_F = \Psi_o \Psi_c
\]
Now bring in the controllability and observability gramians:
\[
L_c := \sum_{0}^{\infty} A^i BB' A'^i \in \mathbb{C}^{n\times n}
\]
\[
L_o := \sum_{0}^{\infty} A'^i C'^i C A^i \in \mathbb{C}^{n\times n}
\]
These matrices are the unique solutions of the Lyapunov equations
\[
L_c = A L_c A' + BB'
\]
\[
L_o = A' L_o A + C'C
\]
It is immediate from the definitions that
\[
[\Psi_c^* \Psi_c] = L_c, \quad [\Psi_o^* \Psi_o] = L_o
\]
Lemma 2  The operator $\Gamma_F^*\Gamma_F$ and the matrix $L_c L_o$ share the same nonzero eigenvalues.

Proof  Let $\mu$ be a nonzero eigenvalue of $\Gamma_F^*\Gamma_F$ and $u$ in $h_2$ a corresponding eigenvector. Thus

$$\Gamma_F^*\Gamma_F u = \mu u$$

This and (4.10) imply

$$\Psi_c^\ast \Psi_o^\ast \Psi_o \Psi_c u = \mu u \quad (4.12)$$

Pre-multiply by $\Psi_c$:

$$\Psi_c \Psi_c^\ast \Psi_o^\ast \Psi_o (\Psi_c u) = \mu (\Psi_c u)$$

This and (4.11) yield

$$L_c L_o (\Psi_c u) = \mu (\Psi_c u)$$

Now $\Psi_c u \neq 0$, from (4.12). Hence $\mu$ is an eigenvalue of $L_c L_o$.

The converse is similar. ■

To proceed we need two extra facts. First, for every operator $\Phi$ on a Hilbert space, $\\|\Phi\\| = (\\|\Phi^*\Phi\\|)^{1/2}$. Secondly, the norm of $\Gamma_F^*\Gamma_F$ equals its maximum eigenvalue. (This is not true in general, but is a consequence of the fact that this operator has finite rank.)

In summary, here is a procedure to compute the norm of the Hankel operator with symbol $\hat{F}$ in $RL_{\infty}$.

Step 1  Compute a minimal realization $C(\lambda - A)^{-1}B$ of $\Pi_1 \hat{F}$.

Step 2  Solve the Lyapunov equations

$$L_c = AL_c A' + BB'$$

$$L_o = A' L_o A + C'C$$

Step 3  Then $\\|\Gamma_F\\|$ equals the square root of the maximum eigenvalue of $L_c L_o$. 

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4.7 The big commutative diagram

The main constructions of this chapter can be summarized in a grand commutative diagram, as shown below. The upper half pertains to the time-domain, the lower half to the frequency-domain. Start with $F$, time-invariant and $h_2$-stable, but not necessarily causal. Let $F_e$ be its time-invariant extension to $l_2$, $\hat{F}$ its tf, $M_\hat{F}$ the multiplication operator, and $\Gamma_F$ and $\Gamma_{\hat{F}}$ the Hankel operators, time- and frequency-domain respectively. The vertical arrows represent the following maps: $h_2 \rightarrow l_2$ and $H_2 \rightarrow L_2$ are subspace insertions; $l_2 \rightarrow h_2^\perp$ and $L_2 \rightarrow H_2^\perp$ are orthogonal projections; and $l_2 \rightarrow L_2$ is $\lambda$-transformation. The outer two curved arrows are restrictions of $\lambda$-transformation. Finally, when $\hat{F}$ is real-rational, $\Psi_c$ and $\Psi_o$ are the controllability and observability operators from a minimal realization of the $H_2^\perp$-component of $\hat{F}$.

4.8 Exercises

1. Suppose

$$\hat{F}(\lambda) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\rho(A) < 1$$

$$L_c := \sum_{i=0}^{\infty} A^i B B' A'^i$$
Prove that
\[ \|\hat{F}\|_2 = \sqrt{\text{trace}\ (DD' + CLcC')}^{1/2} \]

2. Continuing with the previous exercise, prove that \(L_c\) equals the unique solution of the Lyapunov equation
\[ L_c = ALcA' + BB' \]

3. Compute the \(H_2\)-norm of
\[
\begin{bmatrix}
\frac{\lambda^2+1}{\lambda^2} & 1 \\
\frac{\lambda^2}{\lambda^2-\lambda-6} & \lambda
\end{bmatrix}
\]

4. Consider the linear time-invariant causal system with matrix
\[
\begin{bmatrix}
0 & 0 & 0 & \cdots \\
1 & 0 & 0 & \cdots \\
1/2 & 1 & 0 & \cdots \\
1/3 & 1/2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
Is it \(h_2\)-stable?

5. For the set-up in Section 4, prove that \(\|F\| = \|F_c\|\).

6. Consider an \(h_2\)-stable time-invariant system with \(tf\)
\[ \hat{F}(\lambda) = \frac{\lambda^3}{\lambda^2 + 3.5\lambda + 1.5} \]
The roc is the annulus \(.5 < |\lambda| < 3\). Find \([F]\). Compute the distance from \(F\) to the nearest causal \(h_2\)-stable time-invariant system.
Chapter 5

The Continuous-Time Setup

The purpose of this chapter is to collect the continuous analogs of some of the material in Chapters 2-4.

5.1 Time-domain spaces

Consider a signal $x(t)$ defined for all time, $-\infty < t < \infty$, and taking values in $\mathbb{C}^n$. Thus $x$ is a function

$(-\infty, \infty) \mapsto \mathbb{C}^n$

Restrict $x$ to be square-(Lebesgue) integrable:

$$\int_{-\infty}^{\infty} \|x(t)\|^2 dt < \infty$$ (5.1)

The norm in (5.1) is our previously defined norm on $\mathbb{C}^n$. The set of all such signals is the Lebesgue space $L_2(-\infty, \infty)$. (To simplify notation we suppress the dependence of this space on the integer $n$.) This space is a Hilbert space under the inner product

$$< x, y > := \int_{-\infty}^{\infty} x(t)^* y(t) dt$$

Then the norm of $x$, denoted $\|x\|_2$, equals the square-root of the left-hand side of (5.1).

The set of all signals in $L_2(-\infty, \infty)$ that equal zero for almost all $t < 0$ is a closed subspace, denoted $L_2[0, \infty)$. Its orthogonal complement (zero for almost all $t > 0$) is denoted $L_2(-\infty, 0]$.

5.2 Frequency-domain spaces

Consider a function $x(j\omega)$ that is defined for all frequencies, $-\infty < \omega < \infty$, takes values in $\mathbb{C}^n$, and is square-(Lebesgue) integrable with respect to $\omega$. The space of all such functions is
denoted $L_2$ and is a Hilbert space under the inner product
\[ < x, y > := \frac{1}{2\pi} \int_{-\infty}^{\infty} x(j\omega)^* y(j\omega) d\omega \]
The norm on $L_2$ will be denoted $\|x\|_2$. The space $RL_2$, the real-rational functions in $L_2$, consists of $n$-vectors each component of which is real-rational, strictly proper, and without poles on the imaginary axis.

Next, $H_2$ is the space of all functions $x(s)$ that are analytic in Re $s > 0$, take values in $\mathbb{C}^n$, and satisfy the uniform square-integrability condition
\[ \|x\|_2 := \left[ \sup_{\xi > 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \|x(\xi + j\omega)\|^2 d\omega \right]^{1/2} < \infty \]
(We have used the same norm symbol for $L_2(-\infty, \infty)$, $L_2$, and $H_2$. Context determines which is intended.) This makes $H_2$ a Banach space. Functions in $H_2$ are not defined a priori on the imaginary axis, but we can get there in the limit.

**Theorem 1** If $x \in H_2$, then for almost all $\omega$ the limit
\[ \tilde{x}(j\omega) := \lim_{\xi \to 0} x(\xi + j\omega) \]
exists and $\tilde{x}$ belongs to $L_2$. Moreover, the mapping $x \mapsto \tilde{x}$ from $H_2$ to $L_2$ is linear, injective, and norm-preserving.

It is customary to identify $x$ in $H_2$ and its boundary function $\tilde{x}$ in $L_2$. So henceforth we drop the tilde and regard $H_2$ as a closed subspace of the Hilbert space $L_2$. The space $RH_2$ consists of real-rational $n$-vectors that are stable and strictly proper.

The orthogonal complement $H_2^\perp$ of $H_2$ in $L_2$ is the space of functions $x(s)$ with the following properties: $x(s)$ is analytic in Re $s < 0$; $x(s)$ takes values in $\mathbb{C}^n$; the supremum
\[ \sup_{\xi < 0} \int_{-\infty}^{\infty} \|x(\xi + j\omega)\|^2 d\omega \]
is finite. Again, we identify functions in $H_2^\perp$ and their boundary functions in $L_2$.

Now we turn to two Banach spaces. First, an $n \times m$ complex-valued matrix $F(j\omega)$ belongs to the Lebesgue space $L_\infty$ iff $\|F(j\omega)\|$ is essentially bounded (bounded except possibly on a set of measure zero). The norm just used for $F(j\omega)$ is the norm on $\mathbb{C}^{n \times m}$ introduced in Section 2.4 (largest singular value). Then the $L_\infty$-norm of $F$ is defined to be
\[ \|F\|_\infty := \sup_{\omega} \|F(j\omega)\| \]
This makes $L_\infty$ a Banach space. It is easily checked that $F \in RL_\infty$ iff $F$ is real-rational, proper, and without poles on the imaginary axis.

The final space is $H_\infty$. It consists of functions $F(s)$ that are analytic in Re $s > 0$, take values in $\mathbb{C}^{n \times m}$, and are bounded in Re $s > 0$ in the sense that
\[ \sup \{\|F(s)\| : \text{Re } s > 0\} < \infty \]
The left-hand side defines the $H_\infty$-norm of $F$. There is an analog of Theorem 1 in which $H_2$ and $L_2$ are replaced by $H_\infty$ and $L_\infty$ respectively: each function in $H_\infty$ has a unique boundary function in $L_\infty$, and the mapping from $H_\infty$-function to boundary $L_\infty$-function is linear, injective, and norm-preserving. So henceforth we regard $H_\infty$ as a closed subspace of the Banach space $L_\infty$. Finally, $RH_\infty$ consists of those real-rational matrices that are stable and proper.

Let’s recap in the real-rational case:

- $RL_2$: vector-valued, strictly proper, no poles on imaginary axis
- $RH_2$: vector-valued, strictly proper, stable
- $RH^\perp_2$: vector-valued, strictly proper, no poles in $\text{Re } s < 0$
- $RL_\infty$: matrix-valued, proper, no poles on imaginary axis
- $RH_\infty$: matrix-valued, proper, stable.

### 5.3 Connections

This section contains statements of two basic theorems relating the spaces just introduced. The first, a combined Plancherel and Paley-Wiener theorem, connects the time-domain Hilbert spaces and the frequency-domain Hilbert spaces. A mapping from one Hilbert space to another is a **Hilbert space isomorphism** if it is a linear surjection that preserves inner products. (Such a mapping is continuous, preserves norms, is injective, and has a continuous inverse.)

**Theorem 2** The Fourier transform is a Hilbert space isomorphism from $L_2(\mathbb{R})$ onto $H_2$. It maps $L_2[0, \infty)$ onto $H_2^\perp$ and $L_2(-\infty, 0]$ onto $H_2^\perp$.

This important theorem says in particular that $H_2$ is just the set of Laplace transforms of signals in $L_2[0, \infty)$, i.e., of signals on $t \geq 0$ of finite energy.

The second theorem connects the Hilbert space $H_2$ with the Banach spaces $L_\infty$ and $H_\infty$.

**Theorem 3** (i) If $F \in L_\infty$, then $FL_2 \subset L_2$ and

$$
\|F\|_{\infty} = \sup \{ \|Fx\|_2 : x \in L_2, \|x\|_2 = 1 \}
= \sup \{ \|Fx\|_2 : x \in H_2, \|x\|_2 = 1 \}
$$

(ii) If $F \in H_\infty$, then $FH_2 \subset H_2$ and

$$
\|F\|_{\infty} = \sup \{ \|Fx\|_2 : x \in H_2, \|x\|_2 = 1 \}
$$
5.4 Hankel operators

Let’s look at some operators on the spaces we’ve just introduced.

Example 1

The Fourier transform is an operator from $L_2(-\infty, \infty)$ to $L_2$. Theorem 2 says that its norm equals 1.

Example 2

Introduce the direct sum $L_2(-\infty, \infty) = L_2(-\infty, 0] \oplus L_2[0, \infty)$.

Each function $f$ in $L_2(-\infty, \infty)$ has a unique decomposition $f = f_1 + f_2$ with $f_1 \in L_2(-\infty, 0]$ and $f_2 \in L_2[0, \infty)$:

$f_1(t) = f(t), \quad f_2(t) = 0, \quad t \leq 0$

$f_1(t) = 0, \quad f_2(t) = f(t), \quad t > 0$

The function $f \mapsto f_1$ from $L_2(-\infty, \infty)$ to $L_2(-\infty, 0]$ is an operator, the orthogonal projection of $L_2(-\infty, \infty)$ onto $L_2(-\infty, 0]$. It’s easy to prove that its norm equals 1.

In the same way we have $L_2 = H_2^\perp \oplus H_2$.

The orthogonal projection from $L_2$ onto $H_2^\perp$ will be denoted $\Pi_1$ and from $L_2$ onto $H_2$ by $\Pi_2$.

Example 3

Let $F \in L_\infty$ and define the function $\Lambda_F$ from $L_2$ to $L_2$ via

$\Lambda_F g := Fg$

Thus the action of $\Lambda_F$ is multiplication by $F$. Obviously $\Lambda_F$ is linear. Theorem 3 says that $\|\Lambda_F\| = \|F\|_\infty$, so $\Lambda_F$ is bounded. This operator is called a Laurent operator and $F$ is called its symbol; so $\Lambda_F$ is the Laurent operator with symbol $F$.

A related operator is $\Lambda_F|_{H_2}$, the restriction of $\Lambda_F$ to $H_2$, which maps $H_2$ to $L_2$. Theorem 3 says that its norm also equals $\|F\|_\infty$.

Observe that if $F \in H_\infty$ then, also by Theorem 3,

$\Lambda_F H_2 \subset H_2$

The converse is true too: if $\Lambda_F H_2 \subset H_2$, then $F \in H_\infty$.

Example 4

This is the time-domain analog of the previous example. Recall that convolution in the time-domain corresponds to multiplication in the frequency-domain. Suppose $F(s)$ is a matrix-valued function that is analytic in a vertical strip containing the imaginary axis and
that belongs to $L_\infty$. Taking the region of convergence to be this strip, let $f(t)$ denote the inverse bilateral Laplace transform of $F(s)$. Now define the convolution operator $\Xi_f$ from $L_2(-\infty, \infty)$ to $L_2(-\infty, \infty)$ via

$$y = \Xi_f u$$

$$y(t) = \int_{-\infty}^{\infty} f(t - \tau)u(\tau)d\tau$$

This system is linear, but not necessarily causal because $f(t)$ may not equal zero for negative time. Note that the system is causal iff $\Xi_f$ maps $L_2[0, \infty)$ into $L_2[0, \infty)$, i.e., “$\Xi_f$ leaves the future invariant”. The operators $\Xi_f$ and $\Lambda_F$ are intimately related via the Fourier transform. (Draw the commutative diagram.)

**Example 5**

Again let $F \in L_\infty$. The *Toeplitz operator with symbol* $F$, denoted $\Theta_F$, maps $H_2$ to $H_2$ and is defined as follows: for each $g$ in $H_2$, $\Theta_F g$ equals the orthogonal projection of $Fg$ onto $H_2$. Thus

$$\Theta_F = \Pi_2\Lambda_F|H_2$$

As a concrete example consider the scalar-valued function $F(s) = 1/(s - 1)$ in $RL_\infty$. For $g$ in $H_2$ we have

$$Fg = g_1 + g_2$$

$$g_1 \in H_2^1, \ g_2 \in H_2$$

$$g_1(s) = g(1)/(s - 1)$$

$$g_2(s) = [g(s) - g(1)]/(s - 1)$$

Thus $\Theta_F$ maps $g$ to $g_2$.

**Example 6**

For $F$ in $L_\infty$ the *Hankel operator with symbol* $F$, denoted $\Gamma_F$, maps $H_2$ to $H_2^1$ and is defined as

$$\Gamma_F := \Pi_1\Lambda_F|H_2$$

For the example $F(s) = 1/(s - 1)$, $\Gamma_F$ maps $g(s)$ in $H_2$ to $g(1)/(s - 1)$ in $H_2^1$. Note that $\Gamma_F = 0$ if $F \in H_\infty$.

The relationship between the three operators $\Lambda_F$, $\Theta_F$, and $\Gamma_F$ can be described as follows. We have

$$\Lambda_F : H_2^1 \oplus H_2 \to H_2^1 \oplus H_2$$
and correspondingly we can regard $\Lambda F$ as a $2 \times 2$ matrix with operator entries:

$$
\Lambda_F = \begin{bmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{bmatrix}
$$

For example

$$\Lambda_{11} = \Pi_1 \Lambda_F|_{H_2^\perp}$$

It follows from the definitions that $\Lambda_{12} = \Gamma_F$ and $\Lambda_{22} = \Theta_F$. Thus

$$\Lambda_F|_{H_2} = \begin{bmatrix}
\Gamma_F \\
\Theta_F
\end{bmatrix}$$

**Example 7**

In this example we study the Hankel operator with the special symbol

$$F(s) = \begin{bmatrix}
A & B \\
C & 0
\end{bmatrix} := C(s - A)^{-1}B$$

where $A$ is antistable (all eigenvalues in Re $s > 0$). Suppose $A$ is $n \times n$. Such $F$ belongs to $\text{RL}_{\infty}$. The inverse bilateral Laplace transform of $F(s)$ is

$$f(t) = -Ce^{At}B, \quad t < 0$$

$$f(t) = 0, \quad t \geq 0$$

The time-domain analog of the Hankel operator, denoted $\Gamma_F$, maps a function $u$ in $L_2(0, \infty)$ to the function $y$ in $L_2(-\infty, 0]$ defined by

$$y(t) = \int_0^\infty f(t - \tau)u(\tau)d\tau, \quad t < 0$$

$$= -Ce^{At} \int_0^\infty e^{-A\tau}Bu(\tau)d\tau, \quad t < 0 \quad (5.2)$$

Define two auxiliary operators: the *controllability operator*

$$\Psi_c : L_2[0, \infty) \to \mathbb{C}^n$$

$$\Psi_c u := -\int_0^\infty e^{-A\tau}Bu(\tau)d\tau$$

and the *observability operator*

$$\Psi_o : \mathbb{C}^n \to L_2(-\infty, 0]$$

$$(\Psi_ox)(t) := Ce^{At}x, \quad t < 0$$

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From (5.2) we have that
\[ \Gamma_f = \Psi_o \Psi_c \]
There is a systemic interpretation of \( \Gamma_f \) in terms of the usual state-space equations
\[ \dot{x} = Ax + Bu \tag{5.3} \]
\[ y = Cx \tag{5.4} \]
To see the action of \( \Gamma_f \), solve these equations in the following way. First, apply an input \( u \) in \( L_2(0, \infty) \) to equation (5.3) with initial condition \( x(0) = x_0 \) and such that \( x(t) \) is bounded on \( [0, \infty) \). Then
\[ x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau, \quad t \geq 0 \]
so that
\[ x_0 = -\int_0^\infty e^{-A\tau}Bu(\tau)d\tau = \Psi_c u \]
Now solve (5.3) and (5.4) backwards in time starting at \( t = 0 \) and noting that \( u(t) = 0 \) for \( t < 0 \). The solution is
\[ y(t) = Ce^{At}x_0 = (\Psi_o x_0)(t), \quad t < 0 \]
In this way \( \Gamma_f \) maps future input to initial state to past output.

The adjoint of a Laurent operator \( \Lambda_F \) can be obtained explicitly as follows. Introduce the notation
\[ F^\sim(j\omega) := F(j\omega)^* \tag{5.5} \]
If \( F \in RL_\infty \), then we shall interpret \( F^\sim \) as
\[ F^\sim(s) := F(-s)' \]
which is consistent with (5.5). If \( g \) and \( h \) belong to \( L_2 \), then
\[ < \Lambda_F g, h >= < g, \Lambda_F^* h > \]
and
\[ < \Lambda_F g, h > = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(j\omega)^*F(j\omega)^*h(j\omega)d\omega = < g, F^\sim h > \]
We conclude that \( \Lambda_F^* \) equals the Laurent operator with symbol \( F^\sim \).

Similarly, the adjoint of a Hankel operator \( \Gamma_F \) can be shown to be
\[ \Gamma_F^* = \Pi_2\Lambda_F^*|H_2^+ \]
For example if \( F(s) = 1/(s-1) \), then \( \Gamma_F^* \) maps \( h(s) \) in \( H_2^+ \) to \(-h(-1)/s+1) \) in \( H_2 \). (Verify.)

The rank of an operator \( \Phi : \mathcal{X} \to \mathcal{Y} \) is the dimension of the closure of its image space \( \Phi\mathcal{X} \).

Our interest is in Hankel operators with real-rational symbols, Example 7 being a special case.
**Theorem 4** If $F \in RL_{\infty}$, then $\Gamma_F$ has finite rank.

**Proof** There is a unique factorization (by partial-fraction expansion, for example) $F = F_1 + F_2$, where $F_1$ is strictly proper and analytic in $\text{Re } s \leq 0$ and $F_2$ is proper and analytic in $\text{Re } s \geq 0$, i.e., $F_2 \in RH_{\infty}$. Since $\Gamma_F = \Gamma_{F_1}$, we might as well assume at the start that $F$ is strictly proper and analytic in $\text{Re } s \leq 0$. Introduce a minimal realization:

$$F(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

The operator $\Gamma_F$ and its time-domain analog have equal ranks. As in Example 7 the latter operator equals $\Psi_o \Psi_c$. By controllability and observability $\Psi_c$ is surjective and $\Psi_o$ is injective. Hence $\Psi_o \Psi_c$ has rank $n$, so $\Gamma_F$ does too. ■

For the remainder of this section let $F \in RL_{\infty}$. The self-adjoint operator $\Gamma_F^* \Gamma_F$ maps $H_2$ to itself and its rank is finite by Theorem 4. This property guarantees that it does in fact have eigenvalues. We state without proof the following fact.

**Theorem 5** The eigenvalues of $\Gamma_F^* \Gamma_F$ are real and nonnegative and the largest of them equals $\|\Gamma_F\|$.

This theorem implies that $\|\Gamma_F\|$ equals the square root of the largest eigenvalue of $\Gamma_F^* \Gamma_F$. So we could compute $\|\Gamma_F\|$ if we could compute the eigenvalues of $\Gamma_F^* \Gamma_F$. How to do this latter computation is the last topic of this section.

We continue with the notation introduced in Example 7 and the proof of Theorem 4. The self-adjoint operators $\Psi_o \Psi_c^*$ and $\Psi_c^* \Psi_o$ map $\mathbb{C}^n$ to itself. Thus they have matrix representations with respect to the standard basis on $\mathbb{C}^n$. Define the controllability and observability gramians

$$L_c := \int_0^\infty e^{-At} BB' e^{-A't} dt \quad (5.6)$$

$$L_o := \int_0^\infty e^{-A't} C'C e^{-At} dt \quad (5.7)$$

It is routine to show that $L_c$ and $L_o$ are the unique solutions of the Lyapunov equations

$$AL_c + L_c A' = BB' \quad (5.8)$$

$$A'L_o + L_o A = C'C \quad (5.9)$$

**Theorem 6** The operator $\Gamma_F^* \Gamma_F$ and the matrix $L_c L_o$ share the same nonzero eigenvalues.

In summary, the norm of $\Gamma_F$ for $F$ in $RL_{\infty}$ can be computed as follows. First, find a minimal realization $(A, B, C)$ of the antistable part of $F(s)$, i.e.,

$$F(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + (a \text{ matrix in } RH_{\infty})$$

Next, solve the Lyapunov equations (5.8) and (5.9) for $L_c$ and $L_o$. Then $\|\Gamma_F\|$ equals the square-root of the largest eigenvalue of $L_c L_o$. 45
5.5 Nehari’s theorem

In this section we look at the problem of finding the distance from an \( \mathbf{L}_\infty \)-matrix \( \mathbf{R} \) to \( \mathbf{H}_\infty \):

\[
\text{dist} \ (\mathbf{R}, \mathbf{H}_\infty) := \inf \{ \| \mathbf{R} - \mathbf{X} \|_\infty : \mathbf{X} \in \mathbf{H}_\infty \}
\]

In systemic terms we want to approximate, in \( \mathbf{L}_\infty \)-norm, a given unstable transfer matrix by a stable one. Nehari’s theorem is an elegant solution to this problem.

A lower bound for the distance is easily obtained. Fix \( \mathbf{X} \) in \( \mathbf{H}_\infty \). Then

\[
\| \mathbf{R} \circ \mathbf{X} \|_\infty = \| \Lambda \mathbf{R} \circ \Lambda \mathbf{X} \| \geq \| \Pi_1(\Lambda \mathbf{R} - \Lambda \mathbf{X})| \mathbf{H}_2 \| = \| \Gamma \mathbf{R} - \Gamma \mathbf{X} \| = \| \Gamma \mathbf{R} \|
\]

The last equality is due to the fact that \( \Gamma \mathbf{X} = 0 \). Thus \( \| \Gamma \mathbf{R} \| \) is a lower bound for the distance from \( \mathbf{R} \) to \( \mathbf{H}_\infty \). In fact it equals the distance.

**Theorem 7** There exists a closest \( \mathbf{H}_\infty \)-matrix \( \mathbf{X} \) to a given \( \mathbf{L}_\infty \)-matrix \( \mathbf{R} \), and \( \| \mathbf{R} - \mathbf{X} \| = \| \Gamma \mathbf{R} \| \).

In general there are many \( \mathbf{X} \)'s nearest \( \mathbf{R} \). Interpreted in the time-domain Theorem 7 states that the distance from a given noncausal system to the nearest causal one (the systems being linear and time-invariant) equals the norm of the Hankel operator; in other words the norm of the Hankel operator is a measure of noncausality.

5.6 Exercises

1. In the scalar-valued case prove that \( \mathbf{R} \mathbf{L}_2 \) equals the set of all real-rational functions that are strictly proper and have no poles on the imaginary axis.

2. Let

\[
F(s) = \frac{s - 1}{s + 1}
\]

Prove that \( F \mathbf{H}_\infty \) is closed in \( \mathbf{H}_\infty \).

3. Let

\[
F(s) = \frac{s}{s + 1}
\]

Prove that \( F \mathbf{H}_2 \) is not closed in \( \mathbf{H}_2 \).
4. Show that $\Psi_c$ is surjective if $(A, B)$ is controllable and that $\Psi_o$ is injective if $(C, A)$ is observable.

5. Show that the adjoints of $\Psi_c$ and $\Psi_o$ are as follows:

\[
\Psi_c^* : \mathbb{C}^n \rightarrow L_2[0, \infty) \\
(\Psi_c^* x)(t) = -B' e^{-A't} x, \quad t \geq 0 \\
\Psi_o^* : L_2(-\infty, 0] \rightarrow \mathbb{C}^n \\
\Psi_o^* y = \int_{-\infty}^{0} e^{A't} C'y(t) dt
\]

6. Prove that the matrix representations of $\Psi_c \Psi_c^*$ and $\Psi_o^* \Psi_o$ are $L_c$ and $L_o$ respectively.
Chapter 6

Control Systems: Introduction

In this chapter we look at a general control system and some specific examples. All the components in these systems are assumed to be linear, causal, and time-invariant, and are modeled by their transfer functions that are assumed to be proper. Similarly, all signals are in the frequency domain.

The general setup is shown in Figure 1. In this figure \( w, u, z, \) and \( y \) are vector-valued signals: \( w \) is the exogenous input, typically consisting of command signals, disturbances, and sensor noises; \( u \) is the control signal; \( z \) is the output to be controlled, its components typically being tracking errors, filtered actuator signals, etc.; and \( y \) is the measured output. The transfer function \( G \) represents a generalized plant, the fixed part of the system, and \( K \) represents a controller. Partition \( G \) as

\[
G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}
\]

Then Figure 1 stands for the algebraic equations

\[
\begin{align*}
z &= G_{11}w + G_{12}u \\
y &= G_{21}w + G_{22}u \\
u &= Ky
\end{align*}
\]

Figure 1: Basic control system
6.1 Examples

Some common special cases of the general setup are described below.

Example 1: Filtering

\[ x \rightarrow n \rightarrow y \rightarrow K \rightarrow \hat{x} \rightarrow z \]

Figure 2: Filtering example

Figure 2 shows noise \( n \) added to a signal \( x \) to produce an observed signal \( y \), that is passed through a filter \( K \) to yield an estimate \( \hat{x} \) of \( x \). The filtering error is \( z \). The figure can be redrawn as Figure 1 by defining

\[
w = \begin{pmatrix} n \\ x \end{pmatrix}
\]

\[ u = \hat{x} \]

\[
G = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}
\]

Example 2: A feedback system

\[ u_f \rightarrow W_1 \rightarrow K \rightarrow d \rightarrow P \rightarrow W_2 \rightarrow v \]

\[ y \rightarrow K \rightarrow u \rightarrow P \rightarrow W_2 \rightarrow v \]

\[ - \rightarrow W_1 \rightarrow F \rightarrow - \rightarrow W_2 \rightarrow n \]

Figure 3: Single-loop feedback system
Figure 3 shows a standard single-loop feedback system with plant $P$, controller $K$, and feedback sensor $F$. There are two exogenous inputs: a disturbance $d$ and a noise $n$ corrupting the plant output. Also shown are two filters, $W_1$ and $W_2$, generating a filtered control signal, $u_f$, and a filtered plant output, $v$. It is supposed that $u_f$ and $v$ are both to be controlled. To convert to Figure 1 define

$$w = \begin{pmatrix} d \\ n \end{pmatrix}$$

$$z = \begin{pmatrix} v \\ u_f \end{pmatrix}$$

$$G = \begin{bmatrix} \begin{bmatrix} W_2P & 0 \\ 0 & 0 \end{bmatrix} & W_2P \\ -FP & -F \end{bmatrix}$$

**Example 3: A tracking control system**

![Tracking control system diagram](image)

Figure 4: Tracking control system

Shown in Figure 4 is a plant $P$ and a two degree-of-freedom controller, with components $K_1$ and $K_2$. Supposing the signal to be controlled is merely the tracking error between the reference input $w$ and the plant output $v$, we get Figure 1 by defining

$$z = w - v$$

$$y = \begin{pmatrix} w \\ v \end{pmatrix}$$

$$G = \begin{bmatrix} I & -P \\ I & 0 \\ 0 & P \end{bmatrix}$$

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$
Example 4: A model-matching system

Finally, Figure 5 shows a model, $T_1$, that is to be approximated by the cascade of three other systems, $T_2$, $Q$, $T_3$. The three components $T_i$, $i=1,2,3$, are assumed fixed, whereas $Q$ is designable. The signal to be controlled is the error $z$ between the two outputs for a common input $w$. We arrive at Figure 1 by setting

$$G = \begin{bmatrix} T_1 & -T_2 \\ T_3 & 0 \end{bmatrix}$$

$$K = Q$$

Example 5: Bilateral hybrid telerobot

The setup is shown in Figure 6.1. Two robots, a master, $G_m$, and a slave, $G_s$, are controlled by one controller, $K$. A human provides a force command, $f_h$, to the master, while the environment applies a force, $f_e$, to the slave. The controller measures the two velocities, $v_m$ and $v_s$, together with $f_e$ via a force sensor. In turn it provides two force commands, $f_m$ and $f_s$, to the master and slave. Ideally, we want motion following ($v_s = v_m$), a desired master compliance ($v_m$ a desired function of $f_h$), and force reflection ($f_m = f_e$).

We shall design $K$ for two test inputs, namely, $f_e(t)$ is the finite-width pulse

$$f_e(t) = \begin{cases} 
10, & 0 \leq t \leq 0.2 \\
0, & t > 0.2,
\end{cases} \quad (6.1)$$

Figure 5: Model-matching system

Figure 6.1: Bilateral hybrid telerobot.
indicating an abrupt encounter between the slave and a stiff environment, and \( f_h(t) \) is the triangular pulse
\[
f_h(t) = \begin{cases} 
2t, & 0 \leq t \leq 1 \\
-2t + 4, & 1 \leq t \leq 2 \\
0, & t > 2, 
\end{cases}
\] (6.2)
to mimic a ramp-up, ramp-down command.

The generalized error vector \( z \) is taken to have four components: the velocity error \( v_m - v_s \); the compliance error \( f_h - v_m \) (for simplicity, the desired compliance is assumed to be \( v_m = f_h \)); the force-reflection error \( f_m - f_e \); and the slave actuator force. The last component is included as part of regularization, that is, to penalize excessive force applied to the slave. Introducing four scalar weights to be decided later, we arrive at the generalized error vector
\[
\alpha_v (v_m - v_s) \\
\alpha_c (f_h - v_m) \\
\alpha_f (f_m - f_e) \\
\alpha_s f_s
\]
The Laplace transforms of \( f_e \) and \( f_h \) are not rational:
\[
F_e(s) = \frac{10}{s} (1 - e^{-0.2s}) , \quad F_h(s) = \frac{2}{s^2} (1 - e^{-s})^2 .
\]
To get a tractable problem, we shall use second- and third-order Padé approximations,
\[
e^{-Ts} \approx \left[ 1 - \frac{T s}{2} + \frac{(T s)^2}{12} \right] \left[ 1 + \frac{T s}{2} + \frac{(T s)^2}{12} \right]^{-1} \\
e^{-Ts} \approx \left[ 1 - \frac{T s}{2} + \frac{(T s)^2}{10} - \frac{(T s)^3}{120} \right] \left[ 1 + \frac{T s}{2} + \frac{(T s)^2}{10} + \frac{(T s)^3}{120} \right]^{-1} .
\]
Using the third-order approximation for \( F_e(s) \) and the second-order one for \( F_h(s) \), we get
\[
F_e(s) \approx 20 \left[ \frac{0.2}{2} + \frac{0.2^2 s^2}{120} \right] \left[ 1 + \frac{0.2s}{2} + \frac{(0.2s)^2}{10} + \frac{(0.2s)^3}{120} \right]^{-1} =: G_e(s) \\
f_h(s) \approx 2 \left[ \frac{1 + \frac{s}{2} + \frac{s^2}{12} }{2} \right]^{-2} =: G_h(s) .
\]
Incorporating these two prefilters into the preceding block diagram leads to Figure 6.2. The two exogenous inputs \( w_h \) and \( w_e \) are unit impulses. The vector of exogenous inputs is therefore
\[
w = \begin{bmatrix} w_h \\ w_e \end{bmatrix} .
\]
The control system is shown in Figure 6.3, where \( z \) and \( w \) are as above and

\[
y = \begin{bmatrix} v_m \\ v_s \\ f_e \end{bmatrix}, \quad u = \begin{bmatrix} f_m \\ f_s \end{bmatrix}.
\]

Now for the problem. Begin with generic state models for \( G_h, G_m, G_s, G_e \), namely,

\[
\begin{bmatrix} A_h & B_h \\ C_h & 0 \end{bmatrix}, \quad \begin{bmatrix} A_m & B_m \\ C_m & 0 \end{bmatrix}, \quad \begin{bmatrix} A_s & B_s \\ C_s & 0 \end{bmatrix}, \quad \begin{bmatrix} A_e & B_e \\ C_e & 0 \end{bmatrix},
\]

with corresponding states \( x_h, x_m, x_s, x_e \). Use the interconnections in Figure 6.2 and define the overall state

\[
x = \begin{bmatrix} x_m \\ x_s \\ x_e \\ x_h \end{bmatrix}.
\]

In this way get a state model for \( P \).
6.2 Well-posedness

We now wish to define what it means for the system in Figure 1 to be well-posed, and to characterize this condition. For this purpose, introduce two additional inputs, $v_1$ and $v_2$, to get Figure 6.

![Figure 6: Well-posedness problem](image)

Well-posedness means that the nine transfer functions from the three inputs ($w, v_1, v_2$) to the three signals ($z, u, y$) all exist and are proper. Since ($w, u, v_2$) uniquely determine ($z, y$), it suffices to look at the three transfer functions from ($w, v_1, v_2$) to $u$.

**Theorem 1** The system in Figure 1 is well-posed iff the matrix

$$\begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}$$

has a proper inverse.

**Proof** Substitute

$$y = v_2 + G_{21}w + G_{22}u$$

into

$$u = v_1 + Ky$$

to get

$$u = v_1 + K(v_2 + G_{21}w + G_{22}u)$$

Hence

$$(I - KG_{22})u = v_1 + K(v_2 + G_{21}w)$$

Thus the transfer function from $v_1$ to $u$ exists and is proper iff $I - KG_{22}$ has a proper inverse. The latter condition then implies existence and properness of the other two transfer functions from $v_2$ and $w$ to $u$. We conclude that the system is well-posed iff $I - KG_{22}$ has a proper inverse. But this is equivalent to proper invertibility of (6.3).

A useful sufficient condition is provided by the following corollary.

**Corollary 1** The system in Figure 1 is well-posed if $G_{22}$ is strictly proper.

Strict properness of $G_{22}$ means that in Figure 1 there’s attenuation at infinite frequency going around the internal loop from $u$ to $y$ and back to $u$ again.
6.3 Exercises

1. Prove the corollary.

2. Apply the corollary to get sufficient conditions for well-posedness for the four examples in Section 6.1.
Chapter 7

Stability of Feedback Systems

This chapter continues with the general setup introduced in Chapter 6. The main objective is to parametrize all $K$’s that stabilize a given $G$.

7.1 Preliminaries

7.1.1 Controller parametrization

Consider the familiar unity-feedback setup

\[ r \rightarrow e \rightarrow K(s) \rightarrow d \rightarrow u \rightarrow P(s) \]

with the plant and controller being SISO, and with $P(s)$ strictly proper and $K(s)$ proper. There are two equivalent definitions of stability of the feedback system.

1. The first is a state definition. Take two minimal realizations of $P$ and $K$; let the state vectors be $x_P, x_K$. Then $x_P(t)$ and $x_K(t)$ converge to zero as $t$ tends to $\infty$ for every initial condition and with $r = d = 0$. One can test stability by finding the closed-loop $A$-matrix and checking if all its eigenvalues are in $\text{Re} s < 0$.

2. The second is a BIBO definition. For every bounded $r, d$, we have $e, u$ bounded. One can test stability this way: Find coprime polynomials $N, M$ and coprime polynomials $U, V$ such that

\[ P = \frac{N}{M}, \quad K = \frac{U}{V}; \]

then check that the roots of the characteristic polynomial

\[ NU + MV \]  

(7.1)
are in Re s < 0.

Let’s consider the simpler case where $P$ is already stable and we want to characterize all $K$ that maintain feedback stability.

**Lemma 1** Assume $P$ is stable. Then the feedback system is stable iff

$$(\exists Q \in \mathbb{RH}_\infty) K = \frac{Q}{1 - PQ}.$$  

**Proof** ($\implies$) Let $Q$ be the transfer function from $r$ to $u$.

($\impliedby$) We have

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} 1 - PQ & -P(1 - PQ) \\ Q & 1 - PQ \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}. $$

Thus the formula

$$K = \frac{Q}{1 - PQ}, \quad Q \in \mathbb{RH}_\infty$$

parametrizes all controllers that stabilize the feedback loop.

When $P$ is not stable, one could try to stabilize like this: Start with coprime numerator/denominator $P = N/M$; then find $U, V$ to satisfy (7.1); then set $K = U/V$. This won’t quite work, because the resulting $K$ is not guaranteed to be proper. The trick is to take the factors in $\mathbb{RH}_\infty$.

Before we see that, let’s look at Euclid’s algorithm for fun.

### 7.1.2 Euclid’s algorithm

Euclid’s algorithm is a way to find the GCD of two integers, $f, g$. The algorithm produces integers $x, y$ and $d$—the GCD—satisfying the equation

$$fx + gy = d.$$  

We’re interested only in the case where $f, g$ are coprime, in which case $d = 1$.

**Example** The integers $f = 66, g = 35$ are coprime. It follows that there exist integers $x, y$ satisfying the equation

$$fx + gy = 1.$$  

This is called variously a Bezout or Diophantine equation. The construction of $x, y$ (they’re not unique) is as follows: First, divide $g$ into $f$:

$$\frac{66}{35} = 1 + \frac{31}{35}, \quad 66 = 35 + 31.$$
Next, divide 31 into 35:
\[
\frac{35}{31} = 1 + \frac{4}{31}, \quad 35 = 31 + 4.
\]
Keep going until the remainder is 1. Divide 4 into 31:
\[
\frac{31}{4} = 7 + \frac{3}{4}, \quad 31 = 7 \times 4 + 3.
\]
Divide 3 into 4:
\[
\frac{4}{3} = 1 + \frac{1}{3}, \quad 4 = 3 + 1.
\]
Now, backsubstitute; the underlined numbers (remainders) are replaced:
\[
4 - 3 = 1
\]
\[
4 - (31 - 7 \times 4) = 1
\]
\[
4 \times 8 - 31 = 1
\]
\[
(35 - 31) \times 8 - 31 = 1
\]
\[
35 \times 8 - 31 \times 9 = 1
\]
\[
35 \times 8 - (66 - 35) \times 9 = 1
\]
\[
-9 \times 66 + 17 \times 35 = 1
\]
Thus \( x = -9, y = 17 \).

**Example** The algorithm extends to polynomials, for example
\[
f = s^2 + 2s + 1, \quad g = s^3 + 4s^2 + s - 6.
\]
Divide \( f \) into \( g \):
\[
\frac{g}{f} = s + 2 + \frac{-4s - 8}{f}.
\]
Divide \(-4s - 8\) into \( f \):
\[
\frac{f}{-4s - 8} = -\frac{1}{4}s + \frac{1}{-4s - 8}.
\]
Go backwards:
\[
f - (4s + 8)\frac{1}{4}s = 1
\]
\[
f + [g - (s + 2)f]\frac{1}{4}s = 1
\]
\[
f \left(-\frac{1}{4}s^2 - \frac{1}{2}s + 1\right) + g\frac{1}{4}s.
\]
So
\[
x = -\frac{1}{4}s^2 - \frac{1}{2}s + 1, \quad y = \frac{1}{4}s.
\]
7.2 Coprime factorization

Recall that two polynomials $f(s)$ and $g(s)$, with, say, real coefficients, are said to be coprime if their greatest common divisor is 1 (equivalently, they have no common zeros). It follows from Euclid’s algorithm that $f$ and $g$ are coprime iff there exist polynomials $x(s)$ and $y(s)$ such that

$$fx + gy = 1$$  \hspace{1cm} (7.2)

We are going to take the practical route and define two functions $f$ and $g$ in $RH_\infty$ to be coprime (over $RH_\infty$) if there exist $x, y$ in $RH_\infty$ such that (7.2) holds. The more primitive, but equivalent, definition is that $f$ and $g$ are coprime if every common divisor of $f$ and $g$ is invertible in $RH_\infty$, i.e.,

$$h, fh^{-1}, gh^{-1} \in RH_\infty \Rightarrow h^{-1} \in RH_\infty$$

More generally, two matrices $F$ and $G$ in $RH_\infty$ are right-coprime (over $RH_\infty$) if they have equal number of columns and there exist matrices $X$ and $Y$ in $RH_\infty$ such that

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = XF + YG = I$$

This is equivalent to saying that the matrix

$$\begin{bmatrix} F \\ G \end{bmatrix}$$

is left-invertible in $RH_\infty$.

Similarly, two matrices $F$ and $G$ in $RH_\infty$ are left-coprime (over $RH_\infty$) if they have equal number of rows and there exist matrices $X$ and $Y$ in $RH_\infty$ such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} F & G \end{bmatrix} = FX + GY = I$$

Equivalently, $[F \; G]$ is right-invertible in $RH_\infty$.

Now let $G$ be a proper real-rational matrix. A right-coprime factorization of $G$ is a factorization $G = NM^{-1}$ where $N$ and $M$ are right-coprime $RH_\infty$-matrices. Similarly, a left-coprime factorization has the form $G = \tilde{M}^{-1}\tilde{N}$ where $\tilde{N}$ and $\tilde{M}$ are left-coprime. Of course implicit in these definitions is the requirement that $M$ and $\tilde{M}$ be square and nonsingular. We shall require special coprime factorizations, as described in the next lemma.

**Lemma 2** For each proper real-rational matrix $G$ there exist eight $RH_\infty$-matrices satisfying the equations

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$  \hspace{1cm} (7.3)

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I$$  \hspace{1cm} (7.4)

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Equations (7.3) and (7.4) together constitute a *doubly-coprime factorization* of \( G \). It should be apparent that \( N \) and \( M \) are right-coprime and \( \tilde{N} \) and \( \tilde{M} \) are left-coprime; for example, (7.4) implies

\[
[X \quad -Y] \begin{bmatrix} M \\ N \end{bmatrix} = I
\]

proving right-coprimeness.

It’s useful to prove Lemma 1 constructively by deriving explicit formulas for the eight matrices. The formulas use state-space realizations, and hence are readily amenable to computer implementation.

We start with a state-space realization of \( G \),

\[
G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]  \hspace{1cm} (7.5)

where \( A, B, C, D \) are real matrices with \((A, B)\) stabilizable and \((C, A)\) detectable. Now introduce state, input, and output vectors \( x, u, \) and \( y \) respectively so that \( y = Gu \) and

\[
\dot{x} = Ax + Bu
\]  \hspace{1cm} (7.6)

\[
y = Cx + Du
\]  \hspace{1cm} (7.7)

Next, choose a real matrix \( F \) such that \( A_F := A + BF \) is stable, i.e., all eigenvalues in the open left half-plane, and define the vector \( v := u - Fx \) and the matrix \( C_F := C + DF \). Then from (7.6) we get

\[
\dot{x} = A_Fx + Bv
\]

\[
u = Fx + v
\]

\[
y = C_Fx + Dv
\]

Evidently from these equations the tf from \( v \) to \( u \) is

\[
M(s) := \begin{bmatrix} A_F & B \\ F & I \end{bmatrix}
\]  \hspace{1cm} (7.8)

and that from \( v \) to \( y \) is

\[
N(s) := \begin{bmatrix} A_F & B \\ C_F & D \end{bmatrix}
\]  \hspace{1cm} (7.9)

Therefore

\[
u = Mv, \quad y = Nv
\]

so that \( y = NM^{-1}u \), i.e., \( G = NM^{-1} \).
Similarly, by choosing a real matrix $H$ so that $A_H := A + HC$ is stable and defining

$$B_H := B + HD$$

$$\tilde{M}(s) := \begin{bmatrix} A_H & H \\ C & I \end{bmatrix}$$

(7.10)

$$\tilde{N}(s) := \begin{bmatrix} A_H & B_H \\ C & D \end{bmatrix}$$

(7.11)

we get $G = \tilde{M}^{-1}\tilde{N}$. (This can be derived as above by starting with $G'$ instead of $G$.) Thus we’ve obtained four matrices in $\mathbf{RH}_\infty$ satisfying (7.3).

Formulas for the other four matrices to satisfy (7.4) are as follows:

$$X(s) := \begin{bmatrix} A_F & -H \\ C_F & I \end{bmatrix}$$

(7.12)

$$Y(s) := \begin{bmatrix} A_F & -H \\ F & 0 \end{bmatrix}$$

(7.13)

$$\tilde{X}(s) := \begin{bmatrix} A_H & -B_H \\ F & I \end{bmatrix}$$

(7.14)

$$\tilde{Y}(s) := \begin{bmatrix} A_H & -H \\ F & 0 \end{bmatrix}$$

(7.15)

The explanation of where these latter four formulas come from is deferred to Section 7.5.

### 7.3 Stability

We begin this section with a definition of stability for the general setup shown in Figure 1.

![Basic setup](image)

Figure 1: Basic setup
As before, $G$ and $K$ are proper real-rational tf’s. To guarantee well-posedness, we shall assume throughout this chapter that $G_{22}$ is strictly proper. Again, bring in additional inputs as in Figure 2.

![Figure 2: With additional inputs](image)

The system is said to be internally stable, or $K$ stabilizes $G$, provided the nine tf’s from $(w, v_1, v_2)$ to $(z, u, y)$ all belong to $\mathbb{RH}_\infty$. An equivalent definition in terms of state models is this: take minimal state-space realizations of $G$ and $K$ and in Figure 1 set $w = 0$; then $K$ stabilizes $G$ iff the state vectors of $G$ and $K$ converge to zero from every initial condition.

We would like a test for when $K$ stabilizes $G$. Introduce left- and right-coprime factorizations

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$
$$K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$$

Also, introduce two constant matrices,

$$E_u := \begin{bmatrix} 0 \\ I \end{bmatrix}$$

of size $(\dim w + \dim u) \times \dim u$, and

$$E_y := \begin{bmatrix} 0 \\ I \end{bmatrix}$$

of size $(\dim z + \dim y) \times \dim y$.

**Theorem 1** The following are equivalent statements about $K$:

(i) $K$ stabilizes $G$

(ii) $\begin{bmatrix} M & E_uU \\ E_y'N & V \end{bmatrix}$ is invertible in $\mathbb{RH}_\infty$

(iii) $\begin{bmatrix} \tilde{M} & \tilde{N}E_u \\ \tilde{U}E_y' & \tilde{V} \end{bmatrix}$ is invertible in $\mathbb{RH}_\infty$
The idea underlying the equivalence of the first two conditions is simply that the determinant of the matrix in the second is the least common denominator (in $RH_\infty$) of all the tf’s from $w, v_1, v_2$ to $z, u, y$; hence the determinant must be invertible for all these tf’s to belong to $RH_\infty$, and conversely.

The proof of Theorem 1 requires a preliminary result. Insert the factorizations (7.9) into Figure 2, split apart the factors, and introduce two new signals $\xi$ and $\eta$ to get Figure 3.

**Figure 3: For Lemma 2**

**Lemma 3** The nine tf’s in Figure 2 from $w, v_1, v_2$ to $z, u, y$ belong to $RH_\infty$ iff the six tf’s in Figure 3 from $w, v_1, v_2$ to $\xi, \eta$ belong to $RH_\infty$.

**Proof** (If) This direction follows immediately from the equations

$$\begin{bmatrix} z \\ y \end{bmatrix} = N\xi + \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$$

$$u = U\eta + v_1$$

which in turn follow from Figure 3.

(Only if) By right-coprimeness there exist $RH_\infty$-matrices $X$ and $Y$ such that

$$XM + YN = I$$

Hence

$$\xi = XM\xi + YN\xi$$

But from Figure 3.

$$M\xi = \begin{bmatrix} w \\ u \end{bmatrix}$$

$$N\xi = \begin{bmatrix} z \\ y - v_2 \end{bmatrix}$$

Substitution into (7.18) gives

$$\xi = X\begin{bmatrix} 0 \\ u \end{bmatrix} + Y\begin{bmatrix} z \\ y \end{bmatrix} + X\begin{bmatrix} w \\ 0 \end{bmatrix} - Y\begin{bmatrix} 0 \\ v_2 \end{bmatrix}$$
Hence the three transfer matrices from $w, v_1, v_2$ to $\xi$ belong to $\text{RH}_\infty$.

A similar argument works for the remaining three transfer matrices to $\eta$. ■

Proof of Theorem 1  We shall prove the equivalence of the first and second conditions. First, let’s see that the matrix displayed in the second is indeed nonsingular, i.e., its inverse exists as a rational matrix. We have

$$
\begin{bmatrix}
M & E_uU \\
E'_yN & V
\end{bmatrix} = \begin{bmatrix}
I & E_uK \\
E'_yG & I
\end{bmatrix} \begin{bmatrix}
M & 0 \\
0 & V
\end{bmatrix} = \begin{bmatrix}
I & 0 & 0 \\
0 & I & K \\
G_{21} & G_{22} & I
\end{bmatrix} \begin{bmatrix}
M & 0 \\
0 & V
\end{bmatrix} (7.19)
$$

Now

$$
\begin{bmatrix}
M & 0 \\
0 & V
\end{bmatrix}
$$

is nonsingular because both $M$ and $V$ are. Also, since $G_{22}$ is strictly proper, we have that

$$
\begin{bmatrix}
I & 0 & 0 \\
0 & I & K \\
G_{21} & G_{22} & I
\end{bmatrix}
$$

is nonsingular when evaluated at $s = \infty$: its determinant equals 1 there. Thus both matrices on the right-hand side of (7.19) are nonsingular.

The equations corresponding to Figure 3 are

$$
\begin{bmatrix}
M & -E_uU \\
-E'_yN & V
\end{bmatrix} \begin{bmatrix}
\xi \\
\eta
\end{bmatrix} = \begin{bmatrix}
w \\
v_1 \\
v_2
\end{bmatrix}
$$

Thus by Lemma 3 $K$ stabilizes $G$ iff

$$
\begin{bmatrix}
M & -E_uU \\
-E'_yN & V
\end{bmatrix}
$$

is invertible in $\text{RH}_\infty$. But this is equivalent to the second condition. ■

7.4 Stabilizability

Let’s say that $G$ is stabilizable if there exists a (proper real-rational) $K$ that stabilizes it. Not every $G$ is stabilizable; an obvious non-stabilizable $G$ is $G_{12} = 0$, $G_{21} = 0$, $G_{22} = 0$, $G_{11}$ unstable. In this example, the unstable part of $G$ is disconnected from $u$ and $y$. In terms of a state-space model $G$ is stabilizable iff its unstable modes are controllable from $u$ (stabilizability) and observable from $y$ (detectability). The next result is a stabilizability test in terms of left- and right-coprime factorizations

$$
G = NM^{-1} = \tilde{M}^{-1}\tilde{N}
$$
Theorem 2  The following conditions are equivalent:

(i) $G$ is stabilizable

(ii) $M, E'N$ are right-coprime and $M, E_u$ are left-coprime

(iii) $	ilde{M}, 	ilde{N}E_u$ are left-coprime and $	ilde{M}, E'_y$ are right-coprime

The proof requires some preliminaries. The reader will recall the following fact. For each real matrix $F$ there exist real matrices $G$ and $H$ such that

$$F = G \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} H$$

The matrices $G$ and $H$ may be obtained by elementary row and column operations, and the size of the identity matrix equals the rank of $F$. The following analogous result for $\text{RH}_\infty$-matrices is stated without proof.

Lemma 4  For each matrix $F$ in $\text{RH}_\infty$ there exist matrices $G$, $H$, and $F_1$ in $\text{RH}_\infty$ satisfying the equation

$$F = G \begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix} H$$

and having the properties that $G$ and $H$ are invertible in $\text{RH}_\infty$ and $F_1$ is diagonal and nonsingular.

This result is now used to prove the following useful fact that if $M$ and $N$ are right-coprime, then the matrix $\begin{bmatrix} M \\ N \end{bmatrix}$ can be filled out to yield a square matrix that is invertible in $\text{RH}_\infty$.

Lemma 5  Let $M$ and $N$ be $\text{RH}_\infty$-matrices with equal number of columns. Then $M$ and $N$ are right-coprime iff there exist matrices $U$ and $V$ in $\text{RH}_\infty$ such that

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix}$$

is invertible in $\text{RH}_\infty$.

Proof  (If) Define

$$\begin{bmatrix} X & Y \\ ? & ? \end{bmatrix} := \begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1}$$

where a question mark denotes an irrelevant block. Then

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = I$$

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so $M$ and $N$ are right-coprime.

(Only if) Define

$$F := \begin{bmatrix} M \\ N \end{bmatrix}$$

and bring in matrices $G$, $H$, and $F_1$ as per Lemma 4. Since $F$ is left-invertible in $\mathbf{RH}_\infty$ (by right-coprimeness), it follows that

$$\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$$

is left-invertible in $\mathbf{RH}_\infty$ too. But then it must have the form

$$\begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

with $F_1^{-1}$ in $\mathbf{RH}_\infty$. Defining

$$K := G \begin{bmatrix} F_1H & 0 \\ 0 & I \end{bmatrix}$$

we get

$$\begin{bmatrix} M \\ N \end{bmatrix} = K \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Thus the definition

$$\begin{bmatrix} U \\ V \end{bmatrix} := K \begin{bmatrix} 0 \\ I \end{bmatrix}$$

gives the desired result, that

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = K$$

is invertible in $\mathbf{RH}_\infty$. ■

The obvious dual of Lemma 5 is that $M$ and $N$ are left-coprime iff there exist $U$ and $V$ such that

$$\begin{bmatrix} M & N \\ U & V \end{bmatrix}$$

is invertible in $\mathbf{RH}_\infty$.

**Proof of Theorem 2**  We shall prove equivalence of (i) and (ii).

Suppose $G$ is stabilizable. Then by Theorem 1 there exist $U$ and $V$ in $\mathbf{RH}_\infty$ such that

$$\begin{bmatrix} M & E_yU \\ E'_yN & V \end{bmatrix}$$
is invertible in $\text{RH}_\infty$. This implies by Lemma 5 and its dual that $M, E'_yN$ are right-coprime and $M, E_uU$ are left-coprime. But the latter condition implies left-coprimeness of $M, E_u$.

Conversely, assume the second condition in the theorem. Choose, by right-coprimeness and Lemma 5, matrices $X$ and $Y$ in $\text{RH}_\infty$ such that

$$\begin{bmatrix} M & X \\ E'_yN & Y \end{bmatrix}$$

is invertible in $\text{RH}_\infty$. Also, choose, by left-coprimeness, matrices $R$ and $T$ in $\text{RH}_\infty$ such that

$$[M \ E_u] \begin{bmatrix} R \\ T \end{bmatrix} = I \quad (7.20)$$

Now define

$$U := TX \quad (7.21)$$

$$V := Y - E'_yNRX \quad (7.22)$$

Then we have from (7.20), (7.21), and (7.22) that

$$\begin{bmatrix} M & X \\ E'_yN & Y \end{bmatrix} \begin{bmatrix} I & -RX \\ 0 & I \end{bmatrix} = \begin{bmatrix} M & E_uU \\ E'_yN & V \end{bmatrix} \quad (7.23)$$

The two matrices on the left in (7.23) have inverses in $\text{RH}_\infty$, hence so does the matrix on the right.

The next step is to show that $V$ is nonsingular. We have

$$\begin{bmatrix} M & E_uU \\ E'_yN & V \end{bmatrix} = \begin{bmatrix} I & E_uU \\ E'_yG & V \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 \\ 0 & I & U \\ G_{21} & G_{22} & V \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \quad (7.24)$$

Evaluate all the matrices in (7.24) at $s = \infty$; then take determinants of both sides noting that $G_{22}$ is strictly proper and the matrix on the left-hand side of (7.24) is invertible in $\text{RH}_\infty$. This gives

$$0 \neq \det V(\infty)\det M(\infty)$$

Thus $\det V(\infty) \neq 0$, i.e., $V^{-1}$ exists. Hence we can define $K := UV^{-1}$.

Next, note that $U$ and $V$ are right-coprime (this follows from invertibility in $\text{RH}_\infty$ of the matrix on the right-hand side of (7.23)). We conclude from Theorem 1 that $K$ stabilizes $G$.

Hereafter, $G$ will be assumed to be stabilizable. Intuitively, this implies that $G$ and $G_{22}$ share the same unstable poles (counting multiplicities), so to stabilize $G$ it is enough to stabilize $G_{22}$. Let’s define the latter concept explicitly: $K$ stabilizes $G_{22}$ if in Figure 4 the four tf’s from $v_1$ and $v_2$ to $u$ and $y$ belong to $\text{RH}_\infty$. 

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Theorem 3  

The necessity part of the theorem follows from the definitions. To prove sufficiency we need a result analogous to Lemma 3.

Lemma 6  
The four transfer matrices in Figure 4 from $v_1$, $v_2$ to $u$, $y$ belong to $\mathbf{RH}_\infty$ iff the four transfer matrices in Figure 3 from $v_1$, $v_2$ to $\xi$, $\eta$ belong to $\mathbf{RH}_\infty$.

The proof is omitted, it being entirely analogous to that of Lemma 3.

Proof of Theorem 3  
Suppose $K$ stabilizes $G_{22}$. To prove that $K$ stabilizes $G$ it suffices to show, by Lemma 3, that the six transfer matrices in Figure 3 from $w$, $v_1$, $v_2$ to $\xi$, $\eta$ belong to $\mathbf{RH}_\infty$. But by Lemma 6 we know that those from $v_1$, $v_2$ to $\xi$, $\eta$ do. So it remains to show that the two from $w$ to $\xi$, $\eta$ belong to $\mathbf{RH}_\infty$.

Set $v_1 = 0$ and $v_2 = 0$ in Figure 3 and write the corresponding equations:

$$M\xi = \begin{bmatrix} w \\ U\eta \end{bmatrix} \quad (7.25)$$
$$V\eta = E_y'N\xi \quad (7.26)$$

By left-coprimeness there exist matrices $R$ and $T$ in $\mathbf{RH}_\infty$ such that

$$[M \quad E_u] \begin{bmatrix} R \\ T \end{bmatrix} = I \quad (7.27)$$

Post-multiply (7.27) by $\begin{bmatrix} w \\ 0 \end{bmatrix}$ to get

$$MR\begin{bmatrix} w \\ 0 \end{bmatrix} + E_uT\begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} w \\ 0 \end{bmatrix} \quad (7.28)$$

Now subtract (7.28) from (7.25), rearrange, and define

$$\xi_1 := \xi - R\begin{bmatrix} w \\ 0 \end{bmatrix} \quad (7.29)$$
$$v_1 := T\begin{bmatrix} w \\ 0 \end{bmatrix} \quad (7.30)$$
Also, rearrange (7.26) and define

\[ v_2 := E_y' NR \begin{bmatrix} w \\ 0 \end{bmatrix} \]  \hspace{1cm} (7.32)

to get

\[ V\eta = E_y' N\xi_1 + v_2 \]  \hspace{1cm} (7.33)

The block diagram corresponding to (7.31) and (7.33) is Figure 5:

![Block Diagram](image)

Figure 5: For proof of Theorem 3

By Lemma 5 and the fact that \( K \) stabilizes \( G_{22} \) we know that the tf’s in Figure 5 from \( v_1, v_2 \) to \( \xi_1, \eta \) belong to \( \text{RH}_\infty \). But by (7.30) and (7.32) those from \( w \) to \( v_1, v_2 \) belong to \( \text{RH}_\infty \). Hence those from \( w \) to \( \xi_1, \eta \) belong to \( \text{RH}_\infty \). Finally, we conclude from (7.29) that the tf from \( w \) to \( \xi \) belongs to \( \text{RH}_\infty \).

7.5 Parametrization

This section contains a parametrization of all \( K \)’s that stabilize \( G_{22} \). To simplify notation slightly, in this section the subscript 22 on \( G_{22} \) is dropped. The relevant block diagram is Figure 6.

![Block Diagram](image)

Figure 6: Basic loop
Bring in a doubly-coprime factorization of \( G \),
\[
G = NM^{-1} = \tilde{M}^{-1}\tilde{N}
\]
\[
\begin{bmatrix}
\tilde{X} & -\tilde{Y} \\
-\tilde{N} & \tilde{M}
\end{bmatrix}
\begin{bmatrix}
M & Y \\
N & X
\end{bmatrix} = I
\]
(7.34)
and coprime factorizations (not necessarily doubly-coprime) of \( K \),
\[
K = UV^{-1} = \tilde{V}^{-1}\tilde{U}
\]
The first result is analogous to Theorem 1; the proof is omitted.

**Lemma 7** The following are equivalent statements about \( K \):

(i) \( K \) stabilizes \( G \)

(ii) \[
\begin{bmatrix}
M & U \\
N & V
\end{bmatrix}
\]

is invertible in \( RH_{\infty} \)

(iii) \[
\begin{bmatrix}
\tilde{V} & -\tilde{U} \\
-\tilde{N} & \tilde{M}
\end{bmatrix}
\]

is invertible in \( RH_{\infty} \)

The main result of this chapter is the following.

**Theorem 4** The set of all (proper real-rational) \( K \)'s stabilizing \( G \) is parametrized by the formulas
\[
K = (Y - MQ)(X - NQ)^{-1}
\]
(7.35)
\[
= (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M})
\]
(7.36)

\( Q \in RH_{\infty} \)

**Proof** Let’s first prove (7.36). Let \( Q \in RH_{\infty} \). From (7.34) we have
\[
\begin{bmatrix}
I & Q \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\tilde{X} & -\tilde{Y} \\
-\tilde{N} & \tilde{M}
\end{bmatrix}
\begin{bmatrix}
M & Y \\
N & X
\end{bmatrix}
\begin{bmatrix}
I & -Q \\
0 & I
\end{bmatrix} = I
\]
so that
\[
\begin{bmatrix}
\tilde{X} - Q\tilde{N} & -(\tilde{Y} - Q\tilde{M}) \\
-\tilde{N} & \tilde{M}
\end{bmatrix}
\begin{bmatrix}
M & Y - MQ \\
N & X - NQ
\end{bmatrix} = I
\]
(7.37)
Equating the (1,2)-blocks on each side in (7.37) gives
\[
(\tilde{X} - Q\tilde{N})(Y - MQ) = (\tilde{Y} - Q\tilde{M})(X - NQ)
\]
which is equivalent to (7.36).
Next, we show that if $K$ is given by (7.35), it stabilizes $G$. Define

$$U := Y - MQ, \quad V := X - NQ$$

$$\tilde{U} := \tilde{Y} - Q\tilde{M}, \quad \tilde{V} := \tilde{X} - Q\tilde{N}$$

to get from (7.37) that

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = I$$

(7.38)

It follows from (7.38) that $U, V$ are right-coprime and $\tilde{U}, \tilde{V}$ are left-coprime (Lemma 5). Also from (7.38)

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix}$$

is invertible in $\mathbf{RH}_\infty$. So from Lemma 7 $K$ stabilizes $G$.

Finally, suppose $K$ stabilizes $G$. We must show $K$ satisfies (7.35) for some $Q$ in $\mathbf{RH}_\infty$. Let $K = UV^{-1}$ be a right-coprime factorization. From (7.34) and defining $D := \tilde{M}V - \tilde{N}U$ we have

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & \tilde{X}U - \tilde{Y}V \\ 0 & D \end{bmatrix}$$

(7.39)

The two matrices on the left in (7.39) have inverses in $\mathbf{RH}_\infty$, the second by Lemma 7. Hence $D^{-1} \in \mathbf{RH}_\infty$. Define

$$Q := -(\tilde{X}U - \tilde{Y}V)D^{-1}$$

so that (7.39) becomes

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & -QD \\ 0 & D \end{bmatrix}$$

(7.40)

Pre-multiply (7.40) by

$$\begin{bmatrix} M & Y \\ N & X \end{bmatrix}$$

and use (7.34) to get

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} M & Y \\ N & X \end{bmatrix} \begin{bmatrix} I & -QD \\ 0 & D \end{bmatrix}$$

Therefore

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} (Y - MQ)D \\ (X - NQ)D \end{bmatrix}$$
Substitute this into \( K = UV^{-1} \) to get (7.35). ■

As a special case suppose \( G \) is already stable, i.e., \( G \in \textbf{RH}_\infty \). Then in (7.34) we may take

\[
N = \tilde{N} \equiv G \\
\tilde{X} = M = I, \quad X = \tilde{M} = I \\
Y = 0, \quad \tilde{Y} = 0
\]

in which case the formulas in the theorem become simply

\[
K = -Q(I - GQ)^{-1} \\
= -(I - QG)^{-1}Q
\]

There is an interpretation of \( Q \) in this case: \( -Q \) equals the tf from \( v_2 \) to \( u \) in Figure 6.

We can now explain the idea behind the choice of \( X, Y, \tilde{X}, \tilde{Y} \) in Section 7.2. Recall that the state-space equations for \( G \) were

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

that

\[
A_F := A + BF, \quad A_H := A + HC
\]

were stable, and that we defined

\[
B_H := B + HD, \quad C_F := C + DF
\]

Let’s find a stabilizing \( K \) by observer theory. The familiar state-space equations for \( K \) are

\[
\dot{\hat{x}} = A\hat{x} + Bu + H[C\hat{x} + Du - y] \\
u = F\hat{x}
\]

or equivalently

\[
\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}y \\
u = \hat{C}\hat{x}
\]

where

\[
\hat{A} := A + BF + HC + HDF = A_F + HC_F
\]
\[ \hat{B} := -H \]
\[ \hat{C} := F \]

Thus
\[
K(s) = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & 0 \end{bmatrix}
\]

By observer theory \( K \) stabilizes \( G \).

Now find coprime factorizations of \( K \) in the same way as we found coprime factorizations of \( G \) in Section 7.2. To get a right-coprime factorization \( K = YX^{-1} \) we first choose \( \hat{F} \) so that \( \hat{A}_F := \hat{A} + \hat{B}\hat{F} \) is stable. It is convenient to take \( \hat{F} := C_F \), so that \( \hat{A}_F = A_F \). By analogy with (7.7) we get \( K = YX^{-1} \), where

\[
X(s) := \begin{bmatrix} \hat{A}_F & \hat{B} \\ F & I \end{bmatrix}
\]  
\[
= \begin{bmatrix} A_F & -H \\ C_F & I \end{bmatrix}
\]

\[
Y(s) := \begin{bmatrix} \hat{A}_F & \hat{B} \\ \hat{C} & 0 \end{bmatrix}
\]  
\[
= \begin{bmatrix} A_F & -H \\ \hat{C} & 0 \end{bmatrix}
\]

A similar derivation leads to a left-coprime factorization \( K = \bar{X}^{-1}\bar{Y} \), where

\[
\bar{X}(s) := \begin{bmatrix} A_H & -B_H \\ F & I \end{bmatrix}
\]

\[
\bar{Y}(s) := \begin{bmatrix} A_H & -H \\ F & 0 \end{bmatrix}
\]

These formulas coincide with (7.8).

By Lemma 7 we know that
\[
\begin{bmatrix} M & Y \\ N & X \end{bmatrix}
\]

and
\[
\begin{bmatrix} \bar{X} & -\bar{Y} \\ -\bar{N} & \bar{M} \end{bmatrix}
\]

are invertible in \( \mathbf{RH}_\infty \). Hence the product
\[
\begin{bmatrix} \bar{X} & -\bar{Y} \\ -\bar{N} & \bar{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix}
\]

is too. The only surprise is that the product equals the identity matrix, as is verified by algebraic manipulation.
7.6 Closed-loop tf’s

Now we return to the standard setup of Figure 1. Theorem 4 gives every stabilizing $K$ as a transformation of a free parameter $Q$ in $\text{RH}_{\infty}$. The objective in this section is to find the tf from $w$ to $z$ in terms of $Q$.

In the previous section we dropped the subscript on $G_{22}$; now we must restore it. Bring in a doubly-coprime factorization of $G_{22}$:

$$G_{22} = N_{22}M_{22}^{-1} = \tilde{M}_{22}^{-1}\tilde{N}_{22}$$

Then the formula for $K$ is

$$K = (Y_{22} - M_{22}Q)(X_{22} - N_{22}Q)^{-1}$$

$$= (\tilde{X}_{22} - Q\tilde{N}_{22})^{-1}(\tilde{Y}_{22} - Q\tilde{M}_{22})$$

Now define

$$T_1 := G_{11} + G_{12}M_{22}\tilde{Y}_{22}G_{21}$$

$$T_2 := G_{12}M_{22}$$

$$T_3 := \tilde{M}_{22}G_{21}$$

**Theorem 5** The matrices $T_i$ ($i = 1–3$) belong to $\text{RH}_{\infty}$. With $K$ given by (7.42) the transfer matrix from $w$ to $z$ equals $T_1 - T_2QT_3$.

**Proof** The first statement follows from the realizations to be given below. For the second statement we have

$$z = [G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}]w$$

Substitute $G_{22} = N_{22}M_{22}^{-1}$ and (7.43) into $(I - KG_{22})^{-1}$ and use (7.41) to get

$$(I - KG_{22})^{-1} = M_{22}(\tilde{X}_{22} - Q\tilde{N}_{22})$$

Thus from (7.43) again

$$(I - KG_{22})^{-1}K = M_{22}(\tilde{Y}_{22} - Q\tilde{M}_{22})$$

Substitute this into (7.47) and use the definitions of $T_i$ to get $z = (T_1 - T_2QT_3)w$. ■

Compare Theorem 5 with the model-matching example of Chapter 4: the tf from $w$ to $z$ is the same in both cases, namely $T_1 - T_2QT_3$. The conclusion is that, when the controller is parametrized as in (7.42), the general setup reduces to the model-matching setup.
For computations it is useful to have explicit realizations of the transfer matrices $T_i$. Start with a minimal realization of $G$:

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Since the input and output of $G$ are partitioned as

$$\begin{bmatrix} w \\ u \end{bmatrix} \quad \begin{bmatrix} z \\ y \end{bmatrix}$$

the matrices $B$, $C$, and $D$ have corresponding partitions:

$$\begin{bmatrix} B_1 & B_2 \\ C_1 & C_2 \\ D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

Then

$$G_{11}(s) = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \quad G_{12}(s) = \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

$$G_{21}(s) = \begin{bmatrix} A & B_1 \\ C_2 & D_{21} \end{bmatrix} \quad G_{22}(s) = \begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix}$$

Note that $D_{22} = 0$ because $G_{22}$ is strictly proper. It can be proved that stabilizability of $G$ (an assumption from Section 7.4) implies that $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable.

Next, find a doubly-coprime factorization of $G_{22}$ as developed in Section 7.2. For this choose $F$ and $H$ so that

$$A_F := A + B_2 F, \quad A_H := A + H C_2$$

are stable. Then the formulas are as follows:

$$M_{22}(s) = \begin{bmatrix} A_F & B_2 \\ F & I \end{bmatrix}$$

$$N_{22}(s) = \begin{bmatrix} A_F & B_2 \\ C_2 & 0 \end{bmatrix}$$

$$\tilde{M}_{22}(s) = \begin{bmatrix} A_H & H \\ C_2 & I \end{bmatrix}$$

$$\tilde{N}_{22}(s) = \begin{bmatrix} A_H & B_2 \\ C_2 & 0 \end{bmatrix}$$
Finally, substitution into (7.44), (7.45), and (7.46) yields the following realizations:

\[
T_1(s) = \begin{bmatrix}
A & B \\
C & D_{11}
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
A_F & -B_2F \\
0 & A_H
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
B_1 \\
B_1 + HD_{21}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
C_1 + D_{12}F & -D_{12}F
\end{bmatrix}
\]

\[
T_2(s) = \begin{bmatrix}
A_F & B_2 \\
C_1 + D_{12}F & D_{12}
\end{bmatrix}
\]

\[
T_3(s) = \begin{bmatrix}
A_H & B_1 + HD_{21} \\
C_2 & D_{21}
\end{bmatrix}
\]

It can be observed that \(T_i \in \mathbf{RH}_\infty\) as claimed in Theorem 5.

The results of this chapter can be summarized as follows. The matrix \(G\) is assumed to be proper, with \(G_{22}\) strictly proper. Also, \(G\) is assumed to be stabilizable. The formula (7.42) parametrizes all \(K\)'s that stabilize \(G\). In terms of the parameter \(Q\) the tf from \(w\) to \(z\) equals \(T_1 - T_2QT_3\). Such a function of \(Q\) is called affine.

### 7.7 Exercises

1. Prove equivalence of (i) and (iii) in Theorem 1.

2. Prove equivalence of (i) and (iii) in Theorem 2.

3. Suppose \(G_{11} = G_{12} = G_{21} = G_{22}\). Is \(G\) stabilizable?
4. In Figure 6 take \( G(s) = 1/[s(s - 1)] \). Consider a controller of the form
\[
K = \frac{-Q}{1 - GQ}
\]
where \( Q \) is real-rational. Find necessary and sufficient conditions on \( Q \) in order that \( K \) stabilize \( G \).

5. Consider Figure 6 again with both \( G \) and \( K \) strictly proper. Start with a stabilizable, detectable realization of \( G \). Get realizations for the eight matrices in a doubly-coprime factorization of \( G \) as in Section 7.2. Parametrize \( K \) as in Theorem 4. Now reconfigure \( K \) as

![Diagram of feedback system with blocks labeled J and Q]

Derive a realization of \( J \).

6. Continue with the previous exercise as follows. Suppose \( Q \) is strictly proper and stable. Take a minimal realization,
\[
Q(s) = \begin{bmatrix} A_a & B_a \\ C_a & 0 \end{bmatrix}
\]
Define the augmentation
\[
G_a(s) = \begin{bmatrix} A_a & 0 \\ 0 & 0 \end{bmatrix}
\]
and then define the extended plant \( G_e = G + G_a \) (take the state of \( G_e \) to be the stack of the summand states). Show that \( K \) is an observer-based compensator for \( G_e \).

Conclusion: every internal stabilization amounts to adding stable dynamics and then stabilizing the extended plant by an observer-based compensator.

7. Consider a single-loop feedback system with plant \( P \), controller \( C \), negative-unity feedback. Set
\[
P(s) = \frac{s + 1}{(s - 1)(s - 2)}
\]
Parametrize all proper \( C \)'s for which the feedback system is stable. Choose a parameter so that the dc gain from reference input to tracking error equals 0. Compute the resulting \( C \).
Chapter 8

H₂-Optimal Control

This chapter gives two different approaches to an H₂-optimal control problem: one based on the controller parametrization of the previous chapter, and the traditional state-space approach.

8.1 Preliminary example

An LQR example via H₂ optimal control. The problem is

\[ \dot{x} = -x + u, \quad x(0-) = 1 \]

\[ J = \int_{0}^{\infty} (x^2 + u^2) dt. \]

You can check via LQR that the optimal control is \( u =Fx, \quad F = -1/(1 + \sqrt{2}). \)

Move the initial condition to an input:

\[ \dot{x} = -x + u + w, \quad w = \delta. \]

Define a measured output \( y = x \) and a controller output

\[ z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix}. \]

Then \( J = \|z\|_2^2. \)

The block diagram is
where \( P(s) = \frac{1}{s + 1}. \)

Then \( J \) equals the square of the \( H_2 \)-norm from \( w \) to \( z \). Parametrize \( K(s): \)

\[
K = \frac{Q}{1 - PQ}, \quad Q \in \mathbb{RH}_\infty.
\]

We have

\[
T_{zw} = \begin{bmatrix}
P \\
\frac{1 + PK}{PK} \\
\frac{1}{1 + PK}
\end{bmatrix}
= \begin{bmatrix}
P(1 - PQ) \\
PQ
\end{bmatrix}.
\]

Thus

\[
J = \|P(1 - PQ)\|_2^2 + \|PQ\|_2^2
\]

and the problem reduces to

\[
\min_{Q \in \mathbb{RH}_\infty} \|P - P^2Q\|_2^2 + \|PQ\|_2^2.
\]

Bring in the notation \( \tilde{F}(s) = F(-s) \). So for example

\[
\langle FR, Q \rangle = \langle R, \tilde{F}Q \rangle.
\]

Then we get

\[
J = \langle P - P^2Q, P - P^2Q \rangle + \langle PQ, PQ \rangle
\]

\[
J = \langle P, P \rangle - \langle P, P^2Q \rangle - \langle P^2Q, P \rangle + \langle P^2Q, P^2Q \rangle + \langle PQ, PQ \rangle
\]

\[
J = \|P\|^2 - \langle \tilde{P}^2P, Q \rangle - \langle Q, \tilde{P}^2P \rangle + \langle \tilde{P}^2P^2Q, Q \rangle + \langle \tilde{P}PQ, Q \rangle
\]

\[
J = \|P\|^2 - \langle \tilde{P}^2P, Q \rangle - \langle Q, \tilde{P}^2P \rangle + \langle \tilde{P}^2P^2 + \tilde{P}P \rangle Q, Q \rangle.
\]

Let’s try to factor \( \tilde{P}^2P^2 + \tilde{P}P \) as \( \tilde{T}_2T_2 \), with \( T_2 \) stable and minimum phase. This is called spectral factorization. In our example we have

\[
\tilde{P}^2P^2 + \tilde{P}P = \frac{1}{1 - s^2} \left[ \frac{1}{1 - s^2} + 1 \right]
\]

\[
= \frac{1}{1 - s^2} \frac{2 - s^2}{1 - s^2},
\]

so

\[
T_2(s) = \frac{s + \sqrt{2}}{(s + 1)^2}.
\]

So we have

\[
J = \|P\|^2 - \langle \tilde{P}^2P, Q \rangle - \langle Q, \tilde{P}^2P \rangle + \langle T_2Q, T_2Q \rangle.
\]
Now let’s try to write $J$ as

$$J = c + \|T_1 - T_2Q\|_2^2.$$ 

This is completing the square. We require

$$\|P\|^2 - \langle \tilde{P}^2P, Q \rangle - \langle Q, \tilde{P}^2P \rangle + \langle T_2Q, T_2Q \rangle = c + \|T_1\|^2 - \langle \tilde{T}_2T_1, Q \rangle - \langle Q, \tilde{T}_2T_1 \rangle + \langle T_2Q, T_2Q \rangle,$$

so it suffices to take

$$\tilde{P}^2P = \tilde{T}_2T_1,$$

which gives

$$T_1(s) = \frac{1}{(1 + s)(\sqrt{2} - s)}.$$ 

Thus $T_1 \in RL_2$. The projection of $T_1$ onto $RH_2$ is

$$\frac{1}{1 + \sqrt{2}s + 1} =: T_{1a}(s).$$

So now we have

$$J = \text{const.} + \|T_{1a} - T_2Q\|_2^2.$$ 

In this case $T_{1a}/T_2 \in RH_\infty$, so that’s the optimal $Q$:

$$Q(s) = \frac{1}{1 + \sqrt{2} + s}.$$ 

Finally, the optimal $K$ is

$$K = \frac{Q}{1 - PQ} = \frac{1}{1 + \sqrt{2}}.$$ 

Conclusion: We got the optimal state feedback without a Riccati equation. What replaced it is spectral factorization.

### 8.2 Problem statement

We begin with the standard setup:

![Block diagram](image)

It is assumed that $G$ is proper and stabilizable and $G_{22}$ is strictly proper. The input $w$ is standard white noise. The problem is to design a proper $K$ that stabilizes $G$ and minimizes the rms value of $z$, or equivalently, from the continuous analog of Theorem 4.1, minimizes the $H_2$-norm of the transfer function from $w$ to $z$. 

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8.3 Solution via parametrization

In this section only a sketch of the solution is given. The method of solution involves the following steps:

Step 1 Get a doubly-coprime factorization of $G$.

Step 2 Parametrize as in Section 7.5 all proper $K$'s that stabilize $G$. Then the tf from $w$ to $z$ has the form $T_1 - T_2 QT_3$.

Step 3 Choose $Q$ in $\mathbb{RH}_\infty$ to minimize

$$\|T_1 - T_2 QT_3\|_2$$

Step 4 Get $K$ from $Q$ by back-substitution.

The only new step is the third. In view of Example 4 in Chapter 6, we call this the $H_2$-model matching problem: given $T_i \in \mathbb{RH}_\infty$, $i = 1, 2, 3$, find $Q$ in $\mathbb{RH}_\infty$ to minimize

$$\|T_1 - T_2 QT_3\|_2$$

We'll solve this first when the $T_i$’s are $1 \times 1$.

8.3.1 Scalar-valued case

We may as well suppose $T_3 = 1$. The problem is

$$\min_{Q \in \mathbb{RH}_\infty} \|T_1 - T_2 Q\|_2$$

where $T_1, T_2$ are given functions in $\mathbb{RH}_\infty$.

For the solution we need the concepts of inner and outer functions. For a real-rational function $F$ define $F^\sim(s) := F(-s)$. Now let $F \in \mathbb{RH}_\infty$. Then $F$ is inner if $F^\sim F = 1$ and outer if it has no zeros in $\text{Re } s > 0$. It is routine to derive this basic fact: Every function in $\mathbb{RH}_\infty$ has an inner-outer factorization, i.e., for every $F$ in $\mathbb{RH}_\infty$ there exist an inner $F_i$ and an outer $F_o$ such that $F = F_i F_o$. Moreover, if $F^{-1}$ is proper and has no poles on the imaginary axis, then $F_o^{-1} \in \mathbb{RH}_\infty$.

For a solution to the model-matching problem we shall assume that $T_2^{-1}$ is proper and analytic on the imaginary axis, i.e., $T_2^{-1} \in \mathbb{RL}_\infty$. The idea of the solution is easy to explain when $T_1 \in \mathbb{RH}_2$. We want to find the distance, in $H_2$-norm, from $T_1$ to the subspace $T_2 \mathbb{RH}_2$, and also a closest function in this subspace. The assumption just made assures that the subspace $T_2 \mathbb{RH}_2$ is closed, and this assures in turn that there exists a closest function. This closest function is unique and is obtained by taking the orthogonal projection from $T_1$ onto $T_2 \mathbb{RH}_2$.

Fix $Q$ in $\mathbb{RH}_\infty$ such that $T_1 - T_2 Q$ belongs to $\mathbb{RH}_2$. Such $Q$ exists, for example,

$$Q(s) \equiv T_1(\infty)/T_2(\infty)$$
Now do an inner-outer factorization
\[ T_2 = U_i U_o \]
By the assumption on \( T_2 \), \( U_o \) is invertible in \( RH_\infty \). Then
\[ \|T_1 - T_2 Q\|_2 = \|U_i (T_1 U_i^\sim - U_o Q)\|_2 \]
Using the following trivial fact
\[ F \in RL_2, U \text{ inner} \Rightarrow \|U F\|_2 = \|F\|_2 \]
we get
\[ \|T_1 - T_2 Q\|_2 = \|T_1 U_i^\sim - U_o Q\|_2 \]
Define
\[ R := T_1 U_i^\sim, \quad X := U_o Q \]
and note that the mapping
\[ Q \mapsto X : RH_\infty \rightarrow RH_\infty \]
is bijective. Also, \( R \in RL_\infty \) and \( R - X \in RL_2 \), so \( R(\infty) = X(\infty) \). Write uniquely
\[ X = R(\infty) + X_1, \quad X_1 \in RH_2 \]
Recall that \( \Pi_1 \) and \( \Pi_2 \) are the orthogonal projections from \( L_2 \) onto, respectively, \( H_2^\perp \) and \( H_2 \). Then
\[ \|T_1 - T_2 Q\|_2^2 = \|R - X\|_2^2 \]
\[ = \|R - R(\infty) - X_1\|_2^2 \]
\[ = \|\Pi_1 [R - R(\infty) - X_1]\|_2^2 + \|\Pi_2 [R - R(\infty) - X_1]\|_2^2 \]
\[ = \|\Pi_1 [R - R(\infty)]\|_2^2 + \|\Pi_2 [R - R(\infty)] - X_1\|_2^2 \]
Conclusion: the optimal \( X_1 \) is
\[ \Pi_2 [R - R(\infty)] \]
so the optimal \( X \) equals
\[ R(\infty) + \Pi_2 [R - R(\infty)] \]
i.e., \( X \) equals the stable proper part of \( R \).

**Theorem 1** Under the assumption that \( T_2^{-1} \) is proper and analytic on the imaginary axis, there exists a unique optimal \( Q \), namely,
\[ Q = U_o^{-1} \{ R(\infty) + \Pi_2 [R - R(\infty)] \} \]
and for this \( Q \) the model-matching error is
\[ \|T_1 - T_2 Q\|_2 = \|\Pi_1 [R - R(\infty)]\|_2 \]
8.3.2 Matrix-valued case

The method is the same but some of the technicalities are harder. The treatment will be somewhat terse.

For a real-rational matrix \( F \) define \( F \sim(s) := F(-s)' \). A matrix \( F \) in \( RH_\infty \) is inner if \( F \sim F = I \) and outer if the rank of \( F(s) \) equals its number of rows at all points in \( \text{Re} \ s > 0 \). Every matrix \( F \) in \( RH_\infty \) has an inner-outer factorization. If \( F(j\omega) \) has constant rank for all \( 0 \leq \omega \leq \infty \), then its outer factor has a right-inverse in \( RH_\infty \).

Assume \( T_2(j\omega) \) and \( T_3(j\omega) \) have constant rank for all \( 0 \leq \omega \leq \infty \). Do inner-outer factorizations:

\[
T_2 = U_i U_o, \quad T_3' = V_i V_o
\]

Then \( U_o \) and \( V_o \) are both right-invertible in \( RH_\infty \).

Necessary and sufficient conditions for the existence of \( Q \) in \( RH_\infty \) such that \( T_1 - T_2 Q T_3 \in RH_2 \) are

\[
\text{Im} \ T_2(\infty) \subset \text{Im} \ T_1(\infty), \quad \text{Ker} \ T_3(\infty) \subset \text{Ker} \ T_1(\infty)
\]

Assume these and fix such \( Q \). Then

\[
T_1 - T_2 Q T_3 = T_1 - U_i U_o Q V_o' V_i'
\]

Define \( X := U_o Q V_i' \). The mapping

\[
Q \mapsto X : RH_\infty \to RH_\infty
\]

is surjective. So the model-matching problem reduces to

\[
\min_{X \in RH_\infty} \| T_1 - U_i X V_i' \|_2
\]

You can check that the matrix

\[
W_1 := \begin{bmatrix}
U_i \\
I - U_i U_i'
\end{bmatrix}
\]

has the property \( W_1 T_1 W_1 = I \). This implies that

\[
\| W_1 G \|_2 = \| G \|_2, \quad G \in RL_2
\]

Hence

\[
\| T_1 - U_i X V_i' \|_2 = \| W_1 (T_1 - U_i X V_i') \|_2
\]

Similarly, with

\[
W_2 := \begin{bmatrix}
V_i \\
I - V_i V_i'
\end{bmatrix}
\]
we get
\[ \| T_1 - U_i X V'_i \|_2 = \| W_1 (T_1 - U_i X V'_i) W'_2 \|_2 \]
But
\[ W_1 U_i = \begin{bmatrix} I & 0 \end{bmatrix} \]
\[ V'_i W'_2 = \begin{bmatrix} I & 0 \end{bmatrix} \]
and
\[ W_1 T_1 W'_2 = \begin{bmatrix} U_i \sim T_1 V'_i \sim t \end{bmatrix} \]
(“?” denotes irrelevant). Defining
\[ R := U_i \sim T_1 V'_i \sim t \]
we get
\[ \| T_1 - U_i X V'_i \|_2^2 = \left\| \begin{bmatrix} R - X & ? \\ ? & ? \end{bmatrix} \right\|_2^2 = \| R - X \|_2^2 + \| ? \|_2^2 + \| ? \|_2^2 + \| ? \|_2^2 \]
We conclude that the unique optimal \( X \) is
\[ X = R(\infty) + \Pi_2 [R - R(\infty)] \]
You can check that \( X \) again equals the stable proper part of \( R \).

**Theorem 2** Under the assumptions that \( T_2(j\omega) \) and \( T_3(j\omega) \) have constant rank for all \( 0 \leq \omega \leq \infty \) and that
\[ \text{Im } T_2(\infty) \subset \text{Im } T_1(\infty), \quad \text{Ker } T_3(\infty) \subset \text{Ker } T_1(\infty) \]
there exists an optimal \( Q \).

Here’s the procedure for computing an optimal \( Q \):

**Step 1** Do inner-outer factorizations:
\[ T_2 = U_i U_o, \quad T'_3 = V_i V_o \]

**Step 2** Compute
\[ R := U_i \sim T_1 V'_i \sim t \]
\[ X = R(\infty) + \Pi_2 [R - R(\infty)] \]
Step 3 Solve the equation
\[ U_o Q V'_o = X \]
for \( Q \) in \( RH_\infty \).

This procedure is at a high level. Considerable work is involved in reducing it to implementable steps.

For the state-space solution we need some preliminaries about Lyapunov and Riccati equations.

### 8.4 Lyapunov equation

The equation
\[ A'X + XA + Q = 0 \]
is called a Lyapunov equation. Here \( A, Q, X \) are all square matrices, say \( n \times n \), with \( Q \) symmetric.

One situation is where \( A \) and \( Q \) are given and the equation is to be solved for \( X \). Existence and uniqueness are easy to establish in principle. Define the linear map
\[ \Phi : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}, \quad \Phi(X) = A'X + AX \]
Then the Lyapunov equation has a solution \( X \) iff \( Q \in \text{Im} \Phi \); if this condition holds, the solution is unique iff \( \Phi \) is injective, hence bijective. It can be shown that
\[ \sigma(\Phi) = \{ \lambda_1 + \lambda_2 : \lambda_1, \lambda_2 \in \sigma(A) \} \]
So the Lyapunov equation has a unique solution iff \( A \) has the property that no two of its eigenvalues add to zero. For example, if \( A \) is stable, the unique solution is
\[ X = \int_0^\infty e^{A't}Qe^{At}dt \]

We’ll be more interested in another situation: where we want to infer stability of \( A \).

**Theorem 3** Suppose \( A, Q, X \) satisfy the Lyapunov equation, \((Q, A)\) is detectable, and \( Q \) and \( X \) are positive semi-definite. Then \( A \) is stable.

**Proof** For a proof by contradiction, suppose \( A \) has some eigenvalue \( \lambda \) with \( \text{Re} \lambda \geq 0 \). Let \( x \) be a corresponding eigenvector. Pre-multiply the Lyapunov equation by \( x^* \) and post-multiply by \( x \) to get
\[ (2\text{Re} \lambda)x^*Xx + x^*Qx = 0 \]
Both terms on the left are \( \geq 0 \). Hence \( x^*Qx = 0 \), which implies that \( Qx = 0 \) since \( Q \geq 0 \).

Thus
\[ \begin{bmatrix} A - \lambda & Q \\ \end{bmatrix} x = 0 \]
By detectability we must have \( x = 0 \), a contradiction. ■
8.5 Riccati equation

Let $A, Q, R$ be real $n \times n$ matrices with $Q$ and $R$ symmetric. Define the $2n \times 2n$ matrix

$$H := \begin{bmatrix} A & R \\ Q & -A' \end{bmatrix}$$

A matrix of this form is called a Hamiltonian matrix.

It is claimed that $\sigma(H)$ is symmetric about the imaginary axis. To prove this, introduce the $2n \times 2n$ matrix

$$J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

having the property $J^2 = -I$. Then

$$J^{-1} H J = - J H J = -H'$$

so $H$ and $-H'$ are similar. Thus $\lambda$ is an eigenvalue iff $-\lambda$ is.

Now assume $H$ has no eigenvalues on the imaginary axis. Then it must have $n$ in $\text{Re } s < 0$ and $n$ in $\text{Re } s > 0$. Thus the two spectral subspaces $X_- (H)$ and $X_+ (H)$ both have dimension $n$. Let’s focus on $X_- (H)$. Finding a basis for it, stacking the basis vectors up to form a matrix, and partitioning the matrix, we get

$$X_- (H) = \text{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1, X_2 \in \mathbb{R}^{n \times n}$. If $X_1$ is nonsingular, i.e. if the two subspaces

$$X_- (H), \quad \text{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

are complementary, we can set $X := X_2 X_1^{-1}$ to get

$$X_- (H) = \text{Im} \begin{bmatrix} I \\ X \end{bmatrix}$$

Notice that $X$ is then uniquely determined by $H$, i.e. $H \mapsto X$ is a function. We shall denote this function by $Ric$ and write $X = Ric(H)$.

To recap, $Ric$ is a function $\mathbb{R}^{2n \times 2n} \to \mathbb{R}^{n \times n}$ that maps $H$ to $X$ where

$$X_- (H) = \text{Im} \begin{bmatrix} I \\ X \end{bmatrix}$$

The domain of $Ric$, denoted $\text{dom } Ric$, consists of Hamiltonian matrices $H$ with two properties, namely, $H$ has no eigenvalues on the imaginary axis and the two subspaces

$$X_- (H), \quad \text{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

are complementary.

Some properties of $X$ are given below.
Lemma 1 Suppose $H \in \text{dom } \text{Ric}$ and $X = \text{Ric}(H)$. Then

(i) $X$ is symmetric

(ii) $X$ satisfies the algebraic Riccati equation

$A'X + XA + XR^X - Q = 0$

(iii) $A + RX$ is stable

Proof (i) Let $X_1, X_2$ be as above. It’s claimed that

$X'_1X_2$ is symmetric

To prove this, note that there exists a stable matrix $H_-$ in $\mathbb{R}^{n \times n}$ such that

$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}H_-$

($H_-$ is a matrix representation of $H|X_-(H)$.) Pre-multiply this equation by

$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' J$

to get

$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' JH\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' J\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}H_-$

Now $JH$ is symmetric; hence so is the left-hand side of (8.2); hence so is the right:

$(-X'_1X_2 + X'_2X_1)H_- = H'_-( -X'_1X_2 + X'_2X_1)$

$= -H'_-( -X'_1X_2 + X'_2X_1)$

This is a Lyapunov equation. Since $H_-$ is stable, the unique solution is

$-X'_1X_2 + X'_2X_1 = 0$

This proves (8.1).

We have $XX_1 = X_2$. Pre-multiply by $X'_1$ and then use (8.1) to get that $X'_1XX_1$ is symmetric. Since $X_1$ is nonsingular, this implies that $X$ is symmetric too.

(ii) Start with the equation

$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}H_-$

and extract $X_1$ to get

$H\begin{bmatrix} I \\ X \end{bmatrix}X_1 = \begin{bmatrix} I \\ X \end{bmatrix}X_1H_-$
Post-multiply by $X_1^{-1}$:

$$
H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} X_1 H X_1^{-1}
$$

(8.3)

Now pre-multiply by $[X \quad -I]$:

$$
[X \quad -I] H \begin{bmatrix} I \\ X \end{bmatrix}
$$

This is precisely the Riccati equation.

(iii) Pre-multiply (8.3) by $[I \quad 0]$ to get

$$
A + RX = X_1 H X_1^{-1}
$$

Thus $A + RX$ is stable because $H_-$ is. ■

The following result gives verifiable conditions under which $H$ belongs to $\text{dom} \ Ric$.

**Theorem 4** Suppose $H$ has the form

$$
H = \begin{bmatrix} A & -BB' \\ -C'C & -A' \end{bmatrix}
$$

with $(A, B)$ stabilizable and $(C, A)$ detectable. Then $H \in \text{dom} \ Ric$ and $\text{Ric}(H) \geq 0$. If $(C, A)$ is observable, then $\text{Ric}(H) > 0$.

**Proof** We’ll first show that $H$ has no imaginary eigenvalues. Suppose, on the contrary, that $j\omega$ is an eigenvalue and $\begin{pmatrix} x \\ z \end{pmatrix}$ a corresponding eigenvector. Then

$$
Ax - BB'z = j\omega x
$$

$$
-C'C x - A'z = j\omega z
$$

Re-arrange:

$$
(A - j\omega)x = BB'z
$$

(8.4)

$$
-(A - j\omega)^* z = C'C x
$$

(8.5)

Thus

$$
\langle z, (A - j\omega)x \rangle = \langle z, BB'z \rangle = \|B'z\|^2
$$

$$
-\langle x, (A - j\omega)^* z \rangle = \langle x, C'C x \rangle = \|Cx\|^2
$$

and hence

$$
\langle z, (A - j\omega)x \rangle = \|B'z\|^2
$$

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\[(A - j\omega)x, z \rangle = -\|Cx\|^2\]

Thus \(\langle (A - j\omega)x, z \rangle\) is real and
\[-\|Cx\|^2 = \langle (A - j\omega)x, z \rangle = \|B'z\|^2\]

Therefore \(B'z = 0\) and \(Cx = 0\). So from (8.4) and (8.5)
\[(A - j\omega)x = 0\]
\[(A - j\omega)^*z = 0\]

Combine the last four equations to get
\[z^*[A - j\omega \quad B] = 0\]
\[
\begin{bmatrix}
  A - j\omega \\
  C
\end{bmatrix} x = 0
\]

By stabilizability and detectability it follows that \(x = z = 0\), a contradiction.

Next, we’ll show that \(X_1\) is nonsingular, i.e. \(\text{Ker } X_1 = 0\). First, it is claimed that \(\text{Ker } X_1\) is \(H_-\)-invariant. To prove this, let \(x \in \text{Ker } X_1\). Pre-multiply (8.6) by \([I \quad 0]\) to get
\[AX_1 - BB'X_2 = X_1H_-\] (8.7)

Pre-multiply by \(x'X_2'\), post-multiply by \(x\), and use the fact that \(X_2'X_1\) is symmetric (see (8.1)) to get
\[-x'X_2'BB'X_2x = 0\]

Thus \(B'X_2x = 0\). Now post-multiply (8.7) by \(x\) to get \(X_1H_-x = 0\), i.e. \(H_-x \in \text{Ker } X_1\). This proves the claim.

Now to prove that \(X_1\) is nonsingular, suppose on the contrary that \(\text{Ker } X_1 \neq 0\). Then \(H|\text{Ker } X_1\) has an eigenvalue, \(\lambda\), and a corresponding eigenvector, \(x\):
\[H_-x = \lambda x\] (8.8)
Re \( \lambda < 0, \ 0 \neq x \in \text{Ker} X_1 \)

Pre-multiply (8.6) by \([0 \ 1]\):

\[-C'CX_1 - A'X_2 = X_2H_-\]  \(\text{(8.9)}\)

Post-multiply by \(x\) and use (8.8):

\[(A' + \lambda)X_2x = 0\]

Since \(B'X_2x = 0\) too from above, we have

\[x^*X_2'[A + \overline{\lambda} B] = 0\]

Then stabilizability implies \(X_2x = 0\). But if \(X_1x = 0\) and \(X_2x = 0\), then \(x = 0\), a contradiction. This concludes the proof of complementarity.

Now set \(X := \text{Ric}(H)\). We’ll show that \(X \geq 0\). The Riccati equation is

\[A'X + XA - XBB'X + C'C = 0\]

or equivalently

\[(A - BB'X)'X + X(A - BB'X) + XBB'X + C'C = 0\]

Noting that \(A - BB'X\) is stable (Lemma 1), we have

\[X = \int_{0}^{\infty} e^{(A - BB'X)t}(XBB'X + C'C)e^{(A - BB'X)t}dt\]  \(\text{(8.10)}\)

Since \(XBB'X + C'C\) is positive semi-definite, so is \(X\).

Finally, suppose \((C, A)\) is observable. We’ll show that if \(x'Xx = 0\), then \(x = 0\); thus \(X > 0\). Pre-multiply (8.10) by \(x'\) and post-multiply by \(x\):

\[x'Xx = \int_{0}^{\infty} \|B'xe^{(A - BB'X)t}\|^2dt + \int_{0}^{\infty} \|Ce^{(A - BB'X)t}x\|^2dt\]

Thus if \(x'Xx = 0\), then \(Xx = 0\) and

\[Ce^{(A - BB'X)t}x = 0, \ \forall t \geq 0\]

But this implies that \(x\) belongs to the unobservable subspace of \((C, A)\) and so \(x = 0\). \(\blacksquare\)
8.6 State-space solution

Our solution is for a special case of the problem stated in Section 1. The realization of the transfer matrix $G$ is taken to be of the form

$$G(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}$$

The following assumptions are made:

(i) $(A, B_1)$ is stabilizable and $(C_1, A)$ is detectable

(ii) $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable

(iii) $D_{12}' [ C_1 \quad D_{12} ] = [ 0 \quad I ]$

(iv) $[ B_1 \quad D_{21} ] D_{21}' = [ 0 \quad I ]$

Assumption (ii) is necessary and sufficient for $G$ to be internally stabilizable. Assumption (i) is for a technical reason: together with (ii) it guarantees that two Hamiltonian matrices ($H_2$ and $J_2$ below) belong to $\text{dom}(Ric)$.

Assumption (iii) means that $C_1 x$ and $D_{12} u$ are orthogonal and that the latter control penalty is nonsingular and normalized. In the conventional LQG setting this means that there is no cross weighting between the state and control input, and that the control weight matrix is the identity. Other nonsingular control weights can easily be converted to this problem with a change of coordinates in $u$. Relaxing the orthogonality condition introduces a few extra terms in the controller formulas.

Finally, assumption (iv) is dual to (iii) and concerns how the exogenous signal $w$ enters $G$: the plant disturbance and the sensor noise are orthogonal, and the sensor noise weighting is normalized and nonsingular. Two additional assumptions that are implicit in the assumed realization for $G$ is that $D_{11} = 0$ and $D_{22} = 0$. Relaxing these assumptions complicates the formulas substantially.

By Theorem 4 the Hamiltonian matrices

$$H_2 := \begin{bmatrix} A & -B_2B_2' \\ -C_1' C_1 & -A' \end{bmatrix}, \quad J_2 := \begin{bmatrix} A' & -C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}$$

belong to $\text{dom}(Ric)$ and, moreover, $X_2 := Ric(H_2)$ and $Y_2 := Ric(J_2)$ are positive semi-definite. Define

$$F_2 := -B_2'X_2, \quad L_2 := -Y_2C_2'$$

$$A_{F_2} := A + B_2F_2, \quad C_{1F_2} := C_1 + D_{12}F_2$$

$$A_{L_2} := A + L_2C_2, \quad B_{1L_2} := B_1 + L_2D_{21}$$
\[ \hat{A}_2 := A + B_2 F_2 + L_2 C_2 \]

\[ G_c(s) := \begin{bmatrix} A_{F_2} & I \\ C_{1F_2} & 0 \end{bmatrix}, \quad G_f(s) := \begin{bmatrix} A_{L_2} & B_{1L_2} \\ I & 0 \end{bmatrix} \]

and let \( T_{zw} \) denote the tf from \( w \) to \( z \).

**Theorem 5** The unique optimal controller is

\[ K_{opt}(s) := \begin{bmatrix} \hat{A}_2 & -L_2 \\ F_2 & 0 \end{bmatrix} \]

Moreover,

\[ \min \| T_{zw} \|^2_2 = \| G_c B_1 \|^2_2 + \| F_2 G_f \|^2_2 \]

The first term in the minimum cost, \( \| G_c B_1 \|^2_2 \), is associated with optimal control with state feedback and the second, \( \| F_2 G_f \|^2_2 \), with optimal filtering. These two norms can easily be computed as follows:

\[ \| G_c B_1 \|^2_2 = \text{trace} \left( B_1' X_2 B_1 \right) \]
\[ A_{F_2}' X_2 + X_2 A_{F_2} + C_{1F_2}' C_{1F_2} = 0 \]
\[ \| F_2 G_f \|^2_2 = \text{trace} \left( F_2 Y_2 F_2' \right) \]
\[ A_{L_2}' Y_2 + Y_2 A_{L_2}' + B_{1L_2} B_{1L_2}' = 0 \]

The controller \( K_{opt} \) has the well-known separation structure: the controller equations can be written as

\[ \dot{x} = A \hat{x} + B_2 u + L_2 (C_2 \hat{x} - y) \]

\[ u = F_2 \hat{x} \]

\( F_2 \) is the optimal feedback gain were \( x \) directly measured; \( L_2 \) is the optimal filter gain; \( \hat{x} \) is the optimal estimate of \( x \).

**Proof of Theorem 5** Let \( K \) be a proper, stabilizing controller. Start with the system equations

\[ \dot{x} = A x + B_1 w + B_2 u \]
\[ z = C_1 x + D_{12} u \]

and define a new control variable, \( v := u - F_2 x \). The equations become

\[ \dot{x} = A_{F_2} x + B_1 w + B_2 v \]
or in the frequency-domain
\[ z = G_c B_1 w + U v \]

This implies that
\[ T_{zw} = G_c B_1 + U T_{vw} \]

You will prove in an exercise the following fact: \( U \) is inner and \( U \sim G_c \) belongs to \( \mathbf{RH}_2^\perp \). This implies that \( G_c B_1 \) and \( U T_{vw} \) are orthogonal matrices in \( \mathbf{RH}_2 \) (\( T_{vw} \) belongs to \( \mathbf{RH}_2 \) by internal stability). So from the previous equation
\[
\|T_{zw}\|_2^2 = \|G_c B_1\|_2^2 + \|T_{vw}\|_2^2
\]

Look at how \( v \) is generated:

Note that \( K \) stabilizes \( G \) iff \( K \) stabilizes the above system (the two closed-loop systems have identical \( A \)-matrices). So

\[
\min_K \|T_{zw}\|_2^2 = \|G_c B_1\|_2^2 + \min_K \|T_{vw}\|_2^2
\]

and therefore the theorem will be proved once we show the following: for the setup in the previous block diagram, the unique optimal controller is

\[
\begin{bmatrix}
A + B_2 F_2 + L_2 C_2 & -L_2 \\
F_2 & 0
\end{bmatrix}
\]

and the minimum value of \( \|T_{vw}\|_2 \) equals \( \|F_2 G_f\|_2 \). Notice in this setup that \( A + B_2 F_2 \) is stable.

By the assignment \( C_1 \leftarrow -F_2 \), the previous statement becomes this: for

\[
G(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & 0 & I \\
C_2 & D_{21} & 0
\end{bmatrix}
\]
with \( A - B_2C_1 \) stable, the unique optimal controller is
\[
\begin{bmatrix}
A - B_2C_1 + L_2C_2 & L_2 \\
\frac{C_1}{0} & \frac{L_2}{C_1}
\end{bmatrix}
\]
and the minimum cost is \( \|C_1G_f\|_2 \).

The dual of the last statement is this: for
\[
G(s) = \begin{bmatrix}
A & B_1 & B_2 \\
\frac{C_1}{0} & D_{12} & I \\
\frac{C_2}{0} & 0 & 0
\end{bmatrix}
\]
with \( A - B_1C_2 \) stable, the unique optimal controller is
\[
\begin{bmatrix}
A + B_2F_2 - B_1C_2 & B_1 \\
\frac{F_2}{0} & 0
\end{bmatrix}
\]
and the minimum cost is \( \|G_cB_1\|_2 \).

To prove this, apply the controller and let \( \hat{x} \) denote its state. Then the system equations are
\[
\dot{x} = Ax + B_1w + B_2u \\
z = C_1x + D_{12}u \\
y = C_2x + w \\
\dot{\hat{x}} = (A + B_2F_2 - B_1C_2)\hat{x} + B_1y \\
u = F_2\hat{x}
\]
so
\[
\dot{\hat{x}} = A\hat{x} + B_2u + B_1(y - C_2\hat{x})
\]
Defining \( e := x - \hat{x} \), we get
\[
\dot{e} = (A - B_1C_2)e
\]
It’s now easy to infer internal stability from stability of \( A + B_2F_2 \) and \( A - B_1C_2 \). For zero initial conditions on \( x, \hat{x} \), we have \( e(t) \equiv 0 \). Hence
\[
u = F_2\hat{x} = F_2\dot{x}
\] (8.11)

For every proper, stabilizing controller the equation
\[
\|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|T_{vw}\|_2^2
\]
is still valid, showing that
\[
\|T_{zw}\|_2 \geq \|G_cB_1\|_2
\]
But for the present controller, (8.11) implies that \( v \equiv 0 \), i.e., \( T_{vw} = 0 \). Thus the present controller is optimal and the minimum cost is \( \|G_cB_1\|_2 \). Finally, for uniqueness it can be shown (an exercise) that the unique solution of \( T_{vw} = 0 \) is the controller above. \( \blacksquare \)
8.7 Exercises

1. Define

\[ T_1(s) = \frac{s + 1}{s + 2}, \quad T_2(s) = \frac{s^2 - s + 1}{(s + 1)^2} \]

Find a function \( Q \) in \( \mathbf{RH}_\infty \) to minimize \( \|T_1 - T_2Q\|_2 \). Compute the minimum norm.

2. Give an interesting (i.e., nontrivial) example of a \( 2 \times 1 \) inner matrix.

3. Consider

\[ \dot{x} = Ax + Bu, \quad x(0) = x_0 \]

with \( A \) stable. Prove true or false: for every \( u \) in \( L_2[0, \infty) \), \( x(t) \) tends to 0 as \( t \) tends to \( \infty \).

4. Suppose \( u \) and \( y \) are scalar-valued signals and the tf from \( u \) to \( y \) is \( 1/s^2 \). For the standard canonical realization \( (A, B, C) \) consider the optimization problem

\[ \min_{u \in F_x} \int_0^\infty \rho y(t)^2 + u(t)^2 \, dt \]

where \( \rho \) is positive. Find the optimal \( F \). Study the eigenvalues of \( A + BF \) as \( \rho \to 0 \) and as \( \rho \to \infty \).

5. Prove that \( U \) is inner and \( U \sim G_c \in \mathbf{RH}_2^\perp \).


7. You know that right half-plane zeros place definite performance limitations on the control of a system. This exercise illustrates this fact in the present context.

Consider the system

\[ \dot{x} = Ax + Bu, \quad x(0) = x_0 \]
\[ z = Cx \]

Then

\[ z = C(s - A)^{-1}x_0 + C(s - A)^{-1}Bu \]

If \( A \) is stable, we might like to see how small we can make \( \|z\|_2 \) by suitable choice of \( u \) in \( \mathbf{H}_\infty \). In particular, we might like to know if \( \|z\|_2 \) can be made arbitrarily small.

Let

\[ C(s - A)^{-1}B = \frac{s - 1}{(s + 2)(s + 3)} \]
Compute
\[ \inf_{u \in H_\infty} \| C(s - A)^{-1}x_0 + C(s - A)^{-1}Bu \|_2 \]
as a function of \( x_0 \). For what values of \( x_0 \) is the infimum equal to zero. Repeat for
\[ C(s - A)^{-1}B = \frac{s + 1}{(s + 2)(s + 3)} \]