Generalized Quantifiers and Plural Constructions

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1. Introduction

Plural constructions (in short, *plurals*) are as prevalent in natural languages as singular constructions (in short, *singulars*). In this respect, natural languages contrast with the usual symbolic languages, e.g., elementary languages or their higher-order extensions. These are *singular languages*, languages with no counterparts of natural language plurals, because they result from regimenting singular fragments of natural languages. But it is commonly thought that the lack of plurals in the usual symbolic languages results in no deficiency in their expressive power. There is no need to add to symbolic languages counterparts of natural language plurals, one might hold, because plurals are more or less devices for abbreviating their singular cousins. For example, ‘Ali and Baba are funny’ and ‘All boys are funny’ are essentially abbreviations of ‘Ali is funny and Baba is funny’ and ‘Every

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1 What I call *elementary languages* are often called *first-order languages*. I avoid this terminology because it suggests contrasts only with higher-order languages. The regimented plural languages discussed in this paper are first-order extensions of elementary languages; they have no higher-order variables, quantifiers, or predicates.

2 Some natural languages (e.g., Chinese, Japanese, or Korean) have neither singulars nor plurals, because they have no grammatical number system. This does not mean that those languages are like the usual symbolic languages in having no counterparts of plurals or that they have no expressions for talking about many things (as such). In languages without a grammatical number system, count nouns (e.g., the Korean *so* ‘cow’) do not take singular or plural forms, and denote one or more things of a given kind (e.g., any one or more cows). See Yi (preprint 2, §2.3).
boy is funny’, respectively. But there are more recalcitrant plurals, plurals that cannot be considered abbreviations of singulars: ‘The scientists who discovered the structure of DNA cooperated’, ‘All the boys cooperated to lift a piano’, etc. So I reject the traditional view of plurals as abbreviation devices, and propose an alternative view that departs from the tradition that one can trace back to Aristotle through Gottlob Frege. Plurals, in my view, are fundamental linguistic devices that enrich our expressive power, and help to extend the limits of our thoughts. They belong to *basic linguistic categories* that complement the categories to which their singular cousins belong, and they have a *distinct semantic function*: plurals are by and large devices for talking about *many* things (as such), whereas singulaires are more or less devices for talking about *one* thing (‘at a time’).³

Two major milestones leading to the view presented above have been laid out by David Kaplan. He established important limitations of elementary languages vis-à-vis natural language plurals several decades ago. He (1966a) showed that ‘most’ is not definable in elementary languages so that the languages do not have paraphrases of such sentences as the following:

(1) Most are funny.
(2) Most boys are funny.

And he showed that it is the same with a plural construction that came to be known as the Geach-Kaplan sentence:⁴

³ See, e.g., Yi (1999; 2002; 2005; 2006) for an account of plurals based on the view presented above.

⁴ Kaplan communicated his proof about (3) to Quine, and Quine (1974, 238f) states the result. See also Boolos (1984, 432f), who presents the proof. For another discussion of the significance of
(3) Some critics admire only one another.

These results highlight the expressive power of plurals. (1)–(3) involve plural constructions, while elementary languages have counterparts of only singular constructions of natural languages. But it is not usual to take the results to show expressive limitations of singular languages. The proof of undefinability of ‘most’ has helped to support the generalized quantifier approach, which adds non-standard quantifiers to elementary languages without changing their singular character. And it is usual to take Kaplan’s proof of the non-elementary character of (3) to show that (3) is a second-order sentence, a sentence that can be paraphrased into higher-order languages with no counterparts of natural language plurals.

In this paper, I ruminate on Kaplan’s results from a different perspective: they pertain to expressive power of plurals and limitations of singulars. I argue that a proper account of the logic and semantics of (1)–(3) requires taking their plural character seriously. To give such an account, I present symbolic languages, plural languages, that result from augmenting elementary languages with refinements of natural language plurals, and paraphrase the sentences into those symbolic languages. Using the paraphrases, we can explain the logic of (1)–(3) by applying a logical system, plural logic, formulated for the richer symbolic languages that extends elementary logic.

2. Expressive Limitations of Elementary Languages

Elementary languages can be taken to have five kinds of primitive expression:

Kaplan’s result about (3), see Almog (1977).
(a) **Singular Constants**: ‘a’, ‘b’, etc.

(b) **Singular Variables**: ‘x’, ‘y’, ‘z’, etc.

(c) **Predicates**

(i) 1-place predicates: ‘B¹’, ‘C¹’, ‘F¹’, etc.

(ii) 2-place predicates: ‘=’, ‘A²’, etc.

(iii) 3-place predicates: ‘G³’, etc.

Etc.

(d) **Boolean Sentential Connectives**: ‘¬’, ‘∧’, etc.

(e) **Elementary Quantifiers**: the singular existential ‘∃’ and the singular universal ‘∀’

The constants amount to proper names of natural languages (e.g., ‘Ali’ or ‘Baba’); the variables to singular pronouns (e.g., ‘he’, ‘she’, or ‘it’) as used anaphorically (as in, e.g., ‘A boy loves a girl, and she is happy’); the predicates to verbs or verb phrases in the singular form (e.g., ‘is a boy’, ‘is identical with’, ‘admires’, or ‘gives . . . to —’); and the quantifiers to ‘something’ and ‘everything’.

‘Something is a funny boy’, for example, can be paraphrased by the elementary language sentence ‘∃x[B(x) ∧ F(x)]’, where ‘B’ and ‘F’ are counterparts of ‘is a boy’ and ‘is funny’, respectively. Elementary languages have no quantifier that directly amounts to the determiner ‘every’ in, e.g., ‘Every boy is funny.’ But the determiner can be defined in elementary languages so that ‘Every boy is funny’, for example, can be paraphrased by the universal conditional ‘∀x[B(x) → F(x)]’, which amounts to ‘Everything is such that if it is a boy then it is funny.’

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5 The superscript of a predicate indicates the number of its argument places. I usually omit the subscript if the number is clear from the context.
Say that natural language sentences (e.g., ‘Every boy is funny’) are \textit{elementary}, if they can be paraphrased into elementary languages. Kaplan showed that the plural constructions (1)–(3) are \textit{not} elementary:

(1) Most are funny.
(2) Most boys are funny.
(3) Some critics admire only one another.\textsuperscript{6}

That is, he showed that these cannot be paraphrased into elementary languages.

Kaplan’s work on (1) and (2) is motivated by Rescher (1962),\textsuperscript{7} who discusses two quantifiers tantamount to the two uses of ‘most’ in the sentences. (1) might be taken to abbreviate a sentence in which a covert noun follows ‘most’ (e.g., ‘Most things are funny’). Or one might regard ‘most’ in (1) as a \textit{unary} quantifier, one that, like ‘everything’, can combine with one (1-place) predicate to form a sentence. Rescher proposes to add to elementary languages a new quantifier, ‘\textit{M}’, that corresponds to the ‘most’ in (1) as so construed. Using the quantifier, which came to be known as (Rescher’s) \textit{plurality quantifier},\textsuperscript{8} he paraphrases (1) as follows:

\textsuperscript{6} (3) can be taken to abbreviate ‘There are some critics each one of whom is a critic, and admires something only if it is one of them but is not identical with him- or herself.’

\textsuperscript{7} See also Rescher (2004).

\textsuperscript{8} Rescher says that sentences with the quantifier ‘\textit{M}’ involve “the new mode of \textit{plurality-quantification}’, but calls the quantifier itself “M-quantifier” (1962, 373). Kaplan (1966a; 1966b) calls it “the plurality quantifier”.

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where ‘$F$’ is an elementary language predicate amounting to ‘is funny’. He takes the quantification $Mx\phi(x)$ to say that “the set of individuals for which $\phi$ is true has a greater cardinality than the set for which it is false” (1962, 373). And he states that the quantifier ‘$M$’ cannot be defined in elementary languages (ibid., 374), which means that (1) and the like are not elementary, but without explaining how to prove it. Kaplan (1966a) gives a proof of this thesis. Here is a simple proof:

$$\neg MxF(x) \land Mx[F(x) \lor x=a]$$

is satisfiable in any finite model, but in no infinite model. But elementary languages have no such sentence (for their logic is compact).

The proof of undefinability of ‘most’ in (1) extends to the determiner ‘most’ in (2). For it is

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9 I use lower-case Greek letters as metavariables.

10 To relate the undefinability of ‘$M$’ to the natural language quantifier ‘most’, it is necessary to assume that the former captures the latter. This is a controversial assumption; I think ‘most’ is usually used to mean nearly all, not more than half or a majority (of). Westerståhl [1985] holds that it is an ambiguous expression with two readings, which I doubt. Yiannis Moschovakis, in personal conversation, suggests that a plausible reading of most or nearly all is to take it to mean cofinite. But we can take Rescher’s ‘M’ to correspond to ‘more than half’ or ‘a majority’, and the Rescher-Kaplan result to pertain to this quantifier. Although I do not think ‘most’ means a majority, I will still take ‘most’ to mean this for convenience of exposition in this section. See Appendix.

11 Kaplan (1966a) gives a sketch of proofs of this and other interesting facts about languages with ‘$M$’.

12 A model is said to be finite (or infinite) if its domain is finite (or infinite). See Appendix.

13 Rescher (1962, 374) does not introduce a symbolic counterpart of the use of ‘most’ in (2), but he states that it cannot be defined in elementary languages or even their extensions that result from adding ‘$M$’. Barwise & Cooper (1981, 214f) prove this while noting that Kaplan proved it in
straightforward to define the former in terms of the latter; (1), for example, is equivalent to ‘Most

*things that are identical with themselves are funny.*

The above proof of non-elementarity of (4) uses the fact that elementary language sentences
satisfiable in all finite models are satisfiable in an infinite model as well. One can prove this using
an important feature of the logic of elementary languages, *compactness:*

\[
\text{Compactness of Elementary Logic:}
\]

If some sentences of an elementary language \( \mathcal{L} \) logically imply a sentence of \( \mathcal{L} \), then
there are finitely many sentences among the former that logically imply the latter.

This is an important consequence of the completeness of elementary logic. And it is a useful tool
for proving that a sentence is not elementary: a sentence is not elementary if it is logically implied
by infinitely many elementary sentences but not by any finitely many sentences among them. One
can use this to prove most inexpressibility results discussed in this paper.

Some might take Kaplan’s proof of non-elementarity of (1) to show the expressive power of
mathematical languages. They might argue that the quantifier ‘most’ (or ‘\( M’ \)) is not definable in
elementary languages because it is a mathematical expression so that (1), for example, is a statement
about numbers (or sets): it means that the *number* of funny things is greater than the *number* of non-
funny things.\(^{14}\) If so, they might conclude, the proof merely confirms the well-known limitations of

\(^{14}\) Or that the *cardinality* of the *set* of funny things is greater than the *cardinality* of the *set*
of non-funny things.
A natural number $x$ is said to be a successor of a natural number $y$, if $x = y + 1$. 

(6) differs somewhat from the arithmetical sentence used in Kaplan’s proof. See the sentence (C) in Boolos (1984, 432). But it is straightforward to see that they are logically equivalent.
can be paraphrased into elementary languages with ‘C’ and ‘A’, so can (6) into those with ‘N’ and ‘S’. But (6) cannot. For its negation is equivalent to the (full) second-order mathematical induction principle, which cannot be expressed in elementary languages.¹⁷

Most of those who discuss Kaplan’s proof of non-elementarity of (3) take it to show that this also turns out to be a covert statement of a mathematical fact. It is commonly held that (3) is a second-order sentence that, like (5), has the second-order existential quantifier ‘∃²’. And it is usual to take second-order quantifiers to range over sets (or classes), e.g., sets of critics.¹⁸

But (3) does not imply the existence of a set (or class) of critics. To see this, consider the following sentences:

(7.1) Ezra is a critic, Thomas is a critic, Ezra is not Thomas, Ezra admires only Thomas, and Thomas admires only Ezra.

(7.2) Ezra and Thomas are critics who admire only one another.

We can intuitively see that (7.1) logically implies (7.2), and that (7.2) logically implies (3). So (7.1) must logically imply (3). And (7.1), which has a straightforward elementary language counterpart, does not imply the existence of a set. If so, (3) cannot do so, either.¹⁹

¹⁷ Adding the mathematical induction principle to the other Dedekind-Peano axioms (which are elementary) yields a categorical characterization of arithmetical truths. But one cannot give a categorical characterization of arithmetical truths in an elementary language, which can be proved using the compactness of elementary logic.

¹⁸ See, e.g., Quine (1974, 238f), who he argues that (3) is a sentence about sets or classes of critics while stating Kaplan’s result about the sentence.

¹⁹ For an elaboration of this argument, see Yi (2002, Ch. 1). See also Yi (1999; 2005).
It is also wrong to take Kaplan’s proof to show that (3) is a second-order sentence. Although the proof involves paraphrasing (3) by such a sentence, (5), it is not necessary to do so to prove its non-elementarity. Consider the following series of infinitely many elementary language sentences:

\[(8.1) \quad c_1 \text{ is a critic who admires only } c_2, \text{ and } c_1 \text{ is not } c_2.\]

\[(8.2) \quad c_2 \text{ is a critic who admires only } c_3, \text{ and } c_2 \text{ is not } c_3.\]

\[\ldots\]

\[(8.n) \quad c_n \text{ is a critic who admires only } c_{n+1}, \text{ and } c_n \text{ is not } c_{n+1}.\]

where ‘c_1’, ‘c_2’, ‘c_3’, etc. are different proper names. We can intuitively see that these sentences, taken together, logically imply (3): if they hold, then c_1, c_2, c_3, etc. must be critics who admire only one another. But no finitely many sentences among them logically imply (3): (8.1)–(8.n), for example, do not logically imply that c_{n+1} is a critic who admires nothing but c_1, c_2, \ldots, c_{n+1}. So (3)
cannot be paraphrased into elementary languages, whose logic is compact.  

Now some might argue that (3) is not elementary only because it concerns cases involving infinitely many things, and conclude that elementary languages are powerful enough as long as we do not engage in higher mathematical enterprise, and restrict our discourse to finite domains.  It would be wrong to hold this. There is no elementary language sentence that agrees with (3) even on all finite domains. We can show this by applying basic results of finite model theory (see Appendix).  

3. From Singular Languages to Plural Languages

We have seen that the Geach-Kaplan sentence (3) cannot be paraphrased into elementary languages. This is usually taken to show that it is a second-order sentence with an implicit quantification over sets of critics. But there is no good reason to regard the plural quantifier ‘some’ in the sentence as a second-order quantifier. Although Kaplan’s proof of its non-elementarity proceeds by paraphrasing it by its second-order analogue, we can give alternative, more direct proofs that do not rest on the paraphrase as we have seen. By contrast, there is a clear contrast between the plural constructions involved in (3) and the singular constructions involved in, e.g., (7.1)–(7.2) and (8.1)–(8.n), which

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20 One can use the same argument to show the inexpressibility of many other plural constructions (e.g., ‘There are some things each one of which admires one of them’). See Yi (preprint 1), which argues that logic is not axiomatizable, for an elaboration of the argument. See also Yi (2006, 262).

21 Sol Feferman, I think, made this response to the above proof of non-elementarity of (3).

22 The Appendix also proves that no elementary language sentence agrees with ‘MxF(x)’ on all finite models.
are clearly elementary. So there is a good reason to take the non-elementarity of (3) to show the limitations of singulars vis-à-vis plurals. If so, it would be useful to develop plural extensions of elementary languages to give a theory of the logical relations pertaining to plurals, e.g., those that relate (3) to (7.1)–(7.2) or those that relate (3) to (8.1), (8.2), etc. I have developed such languages and characterized their logic in other publications. The languages result from regimenting basic plural constructions of natural languages, and have natural paraphrases of (3). Let me give a sketch of those symbolic languages, which I call (first-order) plural languages.

Plural languages extend elementary languages by including plural cousins of singular variables, predicates, and quantifiers of elementary languages:

\[(b^*) \text{ plural variables: } 'xs', 'ys', 'zs', \text{ etc.} \]

\[(c^*) \text{ plural predicates: } 'C^1' \text{ (for 'to cooperate')}, \ 'D^2' \text{ (for 'to discover')}, \ 'L^2' \text{ (for 'to lift')}, \ 'W^2' \text{ (for 'to write')}, \text{ etc.} \]

\[(e^*) \text{ plural quantifiers: the existential } '\Sigma' \text{ and the universal } '\Pi' \]

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23 Kaplan’s results do not suffice to show that we cannot accommodate natural language plurals in (singular) higher-order languages. But we can add to his results other results that show this. See, e.g., Yi (1999, 172-4) and (2005, 472-6).

24 That is, symbolic languages that have refinements of natural language plurals while containing elementary languages as their singular fragments.


26 I call the logic of plural languages (or, more precisely, the system of logic I have developed to capture the logic) plural logic.

27 I add ‘s’ to lower-case letters of the Roman alphabet to write plural variables, but plural variables (like their singular cousins) are simple expressions with no components with semantic significance.
Plural variables are refinements of the plural pronoun ‘they’ as used anaphorically (as in ‘Some scientists worked in Britain, and they discovered the structure of DNA’). Plural quantifiers, which bind plural variables, are refinements of ‘some things’ and ‘any things’. And plural predicates are refinements of natural language predicates (e.g., ‘to discover’), which contrast with their singular or plural forms. That is, they can combine with plural terms (e.g., ‘they’) because they have argument places that admit a plural term: the only argument place of ‘C’, the first argument place of ‘D’, the second argument place of ‘H’, etc. (I call such argument places plural argument places, and predicates with plural argument places plural predicates.) Elementary language predicates, by contrast, are refinements of the singular forms of natural language predicates (e.g., ‘is funny’ or ‘admires’), and have no argument place that admits plural terms. So they can combine only with singular terms (i.e., singular constants or variables). (Such predicates are called singular predicates.)

Plural languages have a plural predicate of special significance, the two-place plural predicate ‘H’ (whose second-argument place is plural). Like the elementary language predicate ‘=’, it is a logical predicate. We can use it to define complex plural predicates that result from ‘expanding’ singular predicates. We can define the plural (viz., neutral) expansion π^N of a predicate π as follows:

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28 Plural argument places might be exclusively plural (admitting only plural terms) or neutral (admitting singular terms as well). I think natural language predicates (which must be distinguished from their singular and plural forms) have neutral argument places. Accordingly, the plural argument places of the plural predicates I add into plural languages are neutral predicates.

29 I call argument places that admit only singular terms singular argument places, and predicates with only singular argument places singular predicates.
Def. 1 (Neutral Expansions):

\[ \pi^N(xs) \equiv df \forall y[H(y, xs) \rightarrow \pi(y)], \text{ where } \pi \text{ is a singular predicate.}^{30} \]

Then the neutral expansion ‘\(C^N\)’ of the elementary language counterpart ‘\(C\)’ of ‘is a critic’, for example, amounts to the predicate ‘are critics’ (or, more precisely, ‘to be critics’), which can be taken to abbreviate ‘to be such that any one of them is a critic’.

We can now consider the Geach-Kaplan sentence, (3). We can give a straightforward paraphrase of the sentence into plural languages with ‘\(C\)’ and ‘\(A\)’:

\[ (3^*) \sum xs\{C^N(xs) \land \forall x \forall y[H(x, xs) \land A(x, y) \rightarrow x \neq y \land H(y, xs)]\}.^{31} \]

This amounts to an English sentence that we can take (3) to abbreviate:

Some things are such that they are critics (i.e., any one of them is a critic), and any one of them admires something only if the latter is not the former and is one of them.

Using this paraphrase of (3) while invoking plural logic, we can show that (7.1) implies (3), and that

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30 One can take the subscript ‘\(N\)’ to be an operator that takes a predicate (e.g., ‘\(C\)’) to yield a complex predicate. Although plural languages, like elementary languages, have no complex predicates, we can add to the languages the \(\lambda\)-operator that yields complex predicates. Using the operator, we can formulate Def. 1 as follows:

\[ \pi^N = df \lambda xs \forall y[H(y, xs) \rightarrow \pi(y)]. \]

31 Or \( \Sigma xs\{\forall y[H(y, xs) \rightarrow C(y)] \land \forall x \forall y[H(x, xs) \land A(x, y) \rightarrow x \neq y \land H(y, xs)]\} \).
(3) does not imply the existence of a set of critics.\textsuperscript{32}

4. ‘Most’ and Plural Definite Descriptions\textsuperscript{33}

Consider (1) and (2):

\begin{align*}
(1) & \quad \text{Most are funny.} \\
(2) & \quad \text{Most boys are funny.}
\end{align*}

These sentences involve plurals just as (3) does.\textsuperscript{34} But Rescher’s symbolization of (1), ‘\(MxF(x)\)’, results from reducing (1) into a singular construction. His so-called plurality quantifier ‘\(M\)’ is like the elementary language quantifiers ‘\(\exists\)’ and ‘\(\forall\)’ in that it is a singular quantifier, one that can combine with singular variables (e.g., ‘\(x\)’) but not with plural variables (e.g., ‘\(xs\)’). In his English readings of \(Ma\phi\) (or \(Mx\phi x\)), he suggests the plural character of ‘most’. But this makes his readings incoherent. He sometimes reads it as “For most individuals \(a \ldots \phi a\)” (1962, 373; original italics, my underline), and sometimes as “For most \(x\)’s (of the non-empty domain \(D\)) \(\phi x\)” (1962, 374). The

\textsuperscript{32} Plural logic, the logic of plural languages, is a conservative extension of elementary logic. We can use this fact to show that (3) does not imply the existence of a set. See Yi (2005; 2006).

\textsuperscript{33} The discussion in this section does not assume that ‘most’ means \textit{a majority (of)}, but it applies \textit{mutatis mutandis} to ‘a majority (of)’ as well.

\textsuperscript{34} I consider only the use of ‘most’ as the superlative of ‘many’, not of ‘much’, as in (1)–(2). The quantifier as so used combines only with plurals.
plural “individuals” in the first reading cannot provide the antecedent for the singular variable ‘a’;\(^{35}\) similarly, the apparent plural variable ‘x’s’ in the second (which he uses with no explanation\(^{36}\)) does not match with the singular ‘x’.\(^{37}\)

The problem arises because Rescher works in the incipient generalized quantifier framework,\(^{38}\) which augments elementary languages with non-standard quantifiers without changing their singular character. Those who work in the framework might introduce a quantifier corresponding to the use of ‘most’ in (2), ‘Q\(^{\text{most}}\)’, as a binary quantifier that is on a par with the elementary language quantifiers except that it takes two elementary language formulas. (This is called the Rescher quantifier.) (2) is then paraphrased as follows:

\[
Q^{\text{most}}x(B(x), F(x)).
\]

\(^{35}\) Although the universal quantification ‘\(\forall xFx\)’, for example, is commonly rendered as ‘For all \(x\), \(x\) is funny’, which has the same problem, one can give a reading of ‘\(\forall xFx\)’ that uses only singular constructions: ‘Everything \(x\) is such that \(x\) is funny.’ But it is not the same with ‘\(MxFx\)’.

\(^{36}\) The practice of adding the apostrophe and ‘s’ to singular variables dates back to Russell (1937), and is not uncommon in informal exposition. See, e.g., Russell (\textit{ibid.,} 21 & 23) and Gödel (1947, 519). Incidentally, my notation for plural variables was inspired by Gödel’s talk of the “set of \(x\)’s” operation (\textit{ibid.,} 519).

\(^{37}\) What he means might be “Most things, \(xs\), are such that anything, \(x\), that is one of them (i.e., \(xs\)) is such that \(\varphi x\)” or “There are some things, \(xs\), that are most (of all the things in the domain) such that anything, \(x\), that is one of them (i.e., \(xs\)) is such that \(\varphi x\).” My treatment of ‘most’ yields natural symbolizations of these sentences. See, e.g., (1*) below.

\(^{38}\) Rescher (1962) mentions Mostowski (1957), who introduces generalized quantifiers as (singular) second-order predicates that take elementary language predicates as arguments. For the subsequent development of the generalized quantifier approach, see Barwise & Cooper (1982) and Keenan & Stavi (1986). See also Westerståhl (1989).
The contemporary generalized quantifier theory gives a somewhat different treatment of the determiner ‘most’ in (2). The theory regards it as an operator that takes a singular predicate or common noun (e.g., ‘is a boy’ or ‘boy’) to yield a unary quantifier. Those who take this analysis might augment elementary languages with a symbolic counterpart of such an operator, ‘D\text{most}', to paraphrase (2) as follows:

\[(9^*) \ [D\text{most}(x): B] F(x)\]

where ‘[D\text{most}(x): B]’ is a unary quantifier phrase (a generalized quantifier). Both analyses, like Rescher’s paraphrase of (1), ignore the plural character of ‘most’.

Consequently, they cannot give a proper treatment of some siblings of (1)–(2), such as following:

(10) Most of the boys lifted Bob.

(11) Most of the boys who surrounded Bob lifted Bob.

One cannot paraphrase (10) as follows:

\[(12) \ Q\text{most}(B(x), L(x, b))\]

\[\text{39 See Barwise & Cooper (1982) and Keenan & Stavi (1986). See also Montague (1973).}\]

\[\text{40 Imagine, e.g., a situation where Bob is a piano and someone says “None of the boys can lift Bob alone. But most of them cooperated and managed to lift Bob.”}\]
where ‘b’ and ‘L’ correspond to ‘Bob’ and ‘lifted’, respectively. (12) amounts to ‘Most of the boys individually lifted Bob.’ But this is not equivalent to (10), which is true if most of the boys (none of whom can alone lift Bob) cooperated to lift Bob. It is the same with the counterpart of (12) in the contemporary generalized quantifier theory:

\[(12^*) \quad [D_{\text{most}} x: B] L(x, b).\]

And both analyses meet additional difficulties in coping with (11), which is not equivalent to ‘Most of the boys who each surrounded Bob individually lifted Bob.’

To give a proper account of the logic and semantics of ‘most’, it is necessary to give justice to its plural character. To do so, one might add to basic plural languages a plural quantifier corresponding to ‘most’. But those who embrace plurals as peers of singualrs need not take ‘most’ as a primitive quantifier on a par with the standard singular or plural quantifiers in basic plural languages. Its quantifier uses can be analyzed in terms of its uses as plural predicates.

In plural languages, we can introduce a one-place plural predicate, ‘Most’, and a two-place one, ‘Most’, that relate to the uses of ‘most’ in (1) and (2), respectively. ‘Most’ amounts to ‘They are most (of all the things)’, and ‘Most’ to ‘The former are most of the latter.’ And we can paraphrase (1) into plural languages with ‘Most’ and the singular predicate ‘F’ as follows: 

\[\text{McKay (2006, Ch. 5) takes this approach.}\]

\[\text{The 1-place ‘Most’ can be defined in terms of the 2-place ‘Most’:\n\[\text{Most}^1(xs) =_{df} \Sigma y[T(\zeta, y) \land \text{Most}^2(xs, y)].\]}

Applying this definition to (1*) yields the following:

18
(1*) $\sum_{xs}[\textbf{Most}^1(xs) \land F^N(xs)]$

This amounts to a sentence that (1) can be taken to abbreviate:

There are some things that are most (of all the things), and they (each) are funny.

Similarly, we can paraphrase (2) as follows:

(2*) $\sum_{xs}\{\forall z[H(z, xs) \land B(z)] \land \sum_{ys}[\textbf{Most}^2(ys, xs) \land F^N(ys)]\}$.

This amounts to a sentence that (2) can be taken to abbreviate:

There are some things of which anything is one if and only if it is a boy, and there are some things that are most of the former and are funny.

Now, note that (2) is equivalent to a sentence that involves a plural definite description:

(13) Most of the boys are funny.

$\sum_{xs}[\sum_{ys}[\forall z[H(z, ys) \land \textbf{Most}^2(xs, ys)] \land F^N(xs)]]$. 

19
We can prove the equivalence by giving a natural paraphrases of (13) in plural languages.\(^{43}\)

To do so, it is useful to introduce a counterpart of the plural definite description ‘the boys’ into plural languages. Let ‘<\(x: B(x)\)’ be its plural language counterpart. We can then paraphrase (13) as follows:

\[
(13^*) \quad \Sigma_{ys}[\text{Most}^2(ys, <x: Bx>) \land F^N(ys)]
\]

where ‘<\(x: Bx\)’ takes the widest scope.\(^{44}\) And we can characterize the definite description ‘<\(x: B(x)\)’ to show the logical equivalence between (2) and (13). The plural definite description <\(x: \phi(x)\) (where \(\phi(x)\) is a formula) can be defined as follows:

\[\text{Def. 2 (Plural definite descriptions of the first kind)}\]

\[
\pi(<x: \phi(x)>) \equiv_{dt} \Sigma_{ys} \{\forall z[H(z, xs) \leftrightarrow \phi(z)] \land \pi(xs)\}, \text{where } \pi \text{ is a predicate.}
\]

Applying this definition to (13\(^*\)) yields (2\(^*\)).\(^{45}\) This explains the logical equivalence between (2) and

\[\lambda xs \{\Sigma_{ys}[\text{Most}^2(ys, xs) \land F^N(xs)]\}(<x: Bx>).\]

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\]

\[\lambda xs \{\Sigma_{ys}[\text{Most}^2(ys, xs) \land F^N(xs)]\}(<x: Bx>).\]
quantifier analysis of ‘most’. Moreover, the usual generalized quantifier account considers ‘all’ in ‘All boys are funny’ or ‘All the boys are funny’ a mere variant of ‘every’ to take these sentences to be equivalents of ‘Every boy is funny’, which does not imply (2). But we can treat ‘all’ (which combines with plural forms of count nouns) like ‘most’. In plural languages, we can introduce a two-place plural predicate, ‘\textit{All}’, that amounts to ‘are all of’ in, e.g., ‘They are all of my friends.’ We can then take ‘All boys are funny’ to be equivalent to ‘All (of) the boys are funny’, and paraphrase it as follows:

\[
\sum_{xs}[\textit{Most}(ys, <x: Bx>) \land L(ys, b)]
\]

where ‘\textit{L}’ is a two-place plural predicate amounting to ‘lifted’. This paraphrase of (10) is parallel to (13*) except that it has the primitive predicate ‘\textit{L}’ while its counterpart in (13*) is the neutral expansion of ‘\textit{F}’.47

(13).

Let us turn to (10) and (11). Using the predicate ‘\textit{Most}’ and the plural definite description ‘<x: B(x)>’, we can give a straightforward paraphrase of (1) in plural languages:46

\[
\sum_{xs}[\textit{All}(xs, <x: Bx>) \land F^N(xs)].
\]

And we can define ‘\textit{All}’ using only logical expressions of plural languages as follows:

\[
\textit{All}^\sharp(xs, ys) =_{df} \forall z[H(z, xs) \land H(z, ys)] \quad \text{or} \quad \textit{All}^\sharp =_{df} \lambda xs, ys \forall z[H(z, xs) \land H(z, ys)]
\]

For some things are all of, e.g., my friends if and only if any one of them is one of my friends, and \textit{vice versa}. We can then show that ‘\sum_{xs}[\textit{All}(xs, <x: Bx>) \land F^N(xs)]’ implies (2*) because it is equivalent to ‘\exists x B(x) \land \forall x[B(x) \to F(x)]’ (for ‘\Pi x {\textit{Most}}(xs, xs)’ or ‘Any things are most of themselves’ is analytic).

46 Or \[
\lambda xs \{\sum_{ys}[\textit{Most}^\sharp(ys, xs) \land L(ys, b)]\}(<x: Bx>)).
\]

47 But ‘\textit{L}’ and ‘\textit{F}^N’ are both plural predicates; neutral expansions are plural predicates, although they are defined in terms of singular predicates (e.g., ‘\textit{F}’) together with logical expressions of plural languages (e.g., the logical predicate ‘\Pi’).
(11) requires further analysis. The plural definite description in the sentence, ‘the boys who surrounded Bob’, cannot be analyzed in the same way as that in (10) and (13), ‘the boys’. We can see this by comparing the following:

(14) Something is one of the boys if and only if it is a boy.

(15) Something is one of the boys who surrounded Bob if and only if it is a boy who surrounded Bob.

(14) is a logical truth, which provides the basis for Def. 2. But (15) might be false. So one cannot apply the analysis given above of ‘the boys’ to ‘the boys who surrounded Bob’. Using the analysis, we can show that (14) is a logical truth. So applying the same analysis to ‘the boys who surrounded Bob’ would lead to the wrong conclusion that (15) is also a logical truth.

Now, we can introduce into plural languages plural definite descriptions of another kind, counterparts of the likes of ‘the boys who surrounded Bob’. Using ‘I’ as a plural definite description operator, we can take ‘Ixs[B^N(xs) \& S(xs, b)]’ to symbolize ‘the boys who surrounded Bob’. We can then paraphrase (11) by a sentence that results from replacing ‘<x: Bx>’ in (10*) with ‘Ixs[B^N(xs) \& S(xs, b)]’:

\[(11^*) \sum_{ys}[\text{Most}(ys, Ixs[B^N(xs) \& S(xs, b)]) \& L(ys, b)].\]

---

48 The brackets in, e.g., ‘<x: Bx>’ can be considered another plural definite description operator, and it can be defined in terms of ‘I’ (for ‘the boys’, for example, can be considered an abbreviation of ‘the things of which something is one if and only if it is a boy’.)
This amounts to a sentence that (11) can be taken to abbreviate:

There are some things that are most of the boys who surrounded Bob, and they lifted Bob.

We can characterize the second kind of plural definite descriptions as well in plural languages. To do so, it is useful to use a two-place plural predicate, ‘\(\approx\)’, that amounts to ‘to be the same things as’ (the plural “identity” or sameness predicate). The predicate can be defined as follows:

**Def. 3 (Sameness):**

\[
x \approx y \equivdf \forall z [H(z, x) \leftrightarrow H(z, y)].
\]

Using this predicate, we can define plural definite descriptions of the form \(I x \phi(x)\) as follows:

**Def. 4 (Plural definite descriptions of the second kind)**

\[
\pi(I x \phi(x)) \equivdf \sum x \{ \Pi z [\phi(z) \leftrightarrow z \approx x] \land \pi(x) \}, \quad \text{where } \pi \text{ is a predicate.}
\]

Applying this definition to (11*) yields the plural language counterpart of the following:

There are some things that are most of the things that are the same as some things if and only

\[49\text{ For more on plural definite descriptions, see Yi (2006, §4).}\]

\[50\text{ That is, } \sum x \{ \forall z [S(z, b) \leftrightarrow z \approx x] \land \sum y [Most^2(y, x) \land L(y, b)] \}.\]
*if these surrounded Bob*, and they lifted Bob.

So my analysis shows that this sentence is logically equivalent to (11).

We have seen that the quantifier uses of ‘most’ can be analyzed in terms of its use as a plural predicate. The analyses given above involve implicit definitions of the quantifier uses. It would be useful to formulate the definitions.

Using the predicate ‘**Most**’, we can define two related plural quantifiers, ‘**Q**’ and ‘**Q**’:

Def. 5:

(a) \[ [Q_{1 \text{most}}(xs): \tau] \phi = df \Sigma_{xs} [\text{Most}(xs, \tau) \land \phi] \], where \( \tau \) is a plural term.

(b) \[ [Q_{2 \text{most}}(xs): \pi] \phi = df [Q_{1 \text{most}}(xs): <y: \pi(y)>=] \phi \], where \( \pi \) is a singular predicate.

‘\( Q_{1 \text{most}} \)’ is a quantifier that takes a term (e.g., ‘\(<y: B(y)\>\)’) to yield a plural quantifier phrase (e.g., the counterpart of ‘most of the boys’); and ‘\( Q_{2 \text{most}} \)’ one that takes a singular predicate (e.g., ‘\( B \)’) to yield a related plural quantifier phrase (e.g., the counterpart of ‘most boys’). We can take the ‘most of’ in (10)–(11) to amount to the former, and the ‘most’ in (2) to the latter.

Moreover, we can define the singular quantifiers ‘\( Q_{\text{most}} \)’ and ‘\( D_{\text{most}} \)’ in plural languages:

\[ 51 \]

---

\[ 51 \] In addition to the plural quantifiers that can be defined in terms of ‘**Most**’, the definitions use the \( \lambda \)-operator. It is necessary to use the operator because neutral expansions are defined with regard only to predicates in this paper. But it is straightforward to formulate the definitions without using the operator by extending the definition of neutral expansions to formulas. See Yi (2005; 2006) for a definition of neutral expansions applicable to formulas.
Def. 6:

(a) \[\text{D}^\text{most}(x): \pi(x) \equiv \exists \phi(x) (\lambda x \phi(x)) (\lambda x \psi(x))^N(x)\], where \(\pi\) is a singular predicate.

(b) \[Q^\text{most}_2(\phi(x), \psi(x)) \equiv \exists \phi(x) (\lambda x \phi(x)) (\lambda x \psi(x))^N(x)\].

Applying these definitions to, e.g., (9) and (9\*) yields the plural language counterpart of (2).

The plural language analysis of the singular quantifiers ‘Q^most\*’ and ‘D^most\*’ helps to explain why some sentences involving most (e.g., (2)) can be paraphrased into (singular) generalized quantifier languages while others (e.g., (10)) cannot. (2) can be paraphrased into those languages for the same reason that ‘Some boys are funny’ can be paraphrased into elementary languages. In both sentences, the predicates (or formulas) coordinating with the quantifiers are neutral expansions of singular predicates (or elementary language formulas). (10) and (11), by contrast, involve predicates (or formulas) that cannot be considered neutral expansions. So they cannot be paraphrased into generalized quantifier languages, just as ‘Some boys lifted Bob’ and ‘Some of the boys that surrounded Bob lifted Bob’ cannot be into elementary languages.

And the above analysis of ‘most’ explains the partial validity of the thesis that ‘most’ is a conservative quantifier. The singular determiner ‘D^most\*’ is conservative; that is,

\[\text{Conservativity of ‘D}^\text{most}\*’:\]

\[\text{Or, more precisely, ‘Most things that are boys are things that are funny.’}\]

\[\text{See, e.g., Keenan and Stavi (1986, esp. 274ff), who hold that “all English dets satisfy the Conservativity Universal” (ibid., 253).}\]
[D_{most}(x): \pi] \psi(x) \text{ is logically equivalent to } [D_{most}(x): \pi] (\pi(x) \land \psi(x)).

So (2) and (13), for example, are equivalent to the following:

(2a) Most boys are boys and funny.

(13a) Most of the boys are boys and funny.

Proponents of contemporary generalized quantifier theory make this observation to conclude that ‘most’ is conservative. But it is one thing to say that the singular determiner ‘D_{most}’ is conservative, quite another to say that the natural language ‘most’ is so. (11), for example, is not equivalent to the following:

(11a) Most of the boys who surrounded Bob are boys who surrounded Bob and lifted Bob.

Three of the four boys who surrounded Bob might cooperate to lift it although they failed to surround it without the other boy. So ‘most’ cannot be considered conservative.

If so, what explains the fact that ‘D_{most}’ is conservative? It is conservative because ‘Q_{2}^{most}’ is conservative. And this quantifier is conservative because its definition involves a special kind of plural definite descriptions, those of the form <x: \phi(x)>, not because the plural quantifier underlying it, ‘Q_{1}^{most}’, is conservative. Those definite descriptions, as noted above, have a special logical character that distinguishes them from plural definite descriptions of the other kind, those of the form $\text{I}_{x} \phi(x)$. (14a) below is a logical truth, although (15a) below is not:

26
This difference between the two kinds of definite descriptions explains the corresponding disparity in logical status between (14) and (15), which involve English counterparts of the two kinds of definite descriptions: ‘<x: B(x)>’ (‘the boys’) and ‘Ix[\exists^N(xs) \land S(xs, b)]’ (‘the boys who surrounded Bob’). And it explains why (13) and (13a) are equivalent although (11) and (11a) are not: any things that are most of the boys must be boys (every one of them must be a boy), although it is not the same with the boys who surrounded Bob (some things that are most of them might fail to surround Bob.) And this explanation of the equivalence between (13) and (13a) extends to that between (2) and (2a); (2) and (2a), on my analysis, are abbreviations of (13) and (13a), respectively (see Def. 6(a)).

Keenan and Stavi hold that “Extensional determiners in all languages are always interpreted by conservative functions” (1986, 260; original italics), and formulate this as a “theorem” (ibid., 276). The “theorem”, we have seen, does not reflect the nature and range of natural language determiners. It is an artifact of the constraints that proponents of generalized quantifier theory impose on their symbolic languages, which have serious limitations with regard to plurals. In contrast to the theory, the analysis presented above pays special attention to the plural character of ‘most’, and yields a proper explanation of the conservativity of its singular analogues.

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References


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Appendix

We can use basic results of finite model theory to show the following:

(A) There is no elementary language sentence that agrees with Rescher’s “plurality” quantification \( MxF(x) \) on all finite models.

(B) There is no elementary language sentence that agrees with the Geach-Kaplan sentence (3), ‘Some critics admire only one another’, on all finite domains.

Let \( \mathcal{L}_F \) be the elementary language whose only non-logical expression is ‘\( F \)’, and \( \mathcal{L}_A \) the one whose only non-logical expression is ‘\( A \)’. Then a model \( M \) of \( \mathcal{L}_F \) is a pair \(<D_M, F^M>\) such that \( D_M \) is a non-empty set and \( F^M \) a subset of \( D_M \); and a model \( M \) of \( \mathcal{L}_A \) a pair \(<D_M, A^M>\) such that \( D_M \) is a non-empty
set and $A^M$ a subset of $D^M \times D^M$. Say that a model $M$ is finite, if its domain, $D^M$, is a finite set.\textsuperscript{54} Then we can prove the following:\textsuperscript{55}

\[(A^*) \text{ There is no sentence } \phi \text{ of } \mathcal{L}_F \text{ such that } \phi \text{ holds in a finite model } M \text{ of } \mathcal{L}_F \text{ if and only if } |D^M \setminus F^M| = |F^M|.
\]

But \(\sim Mx(F(x)) \land \sim Mx \sim F(x)\) is such a sentence. So (A) holds.

To formulate a theorem that we can use to show (B), it is useful to use the following notions:

**Definitions:** Let $M (=<D^M, A^M>)$ be a model of $\mathcal{L}_A$. Then

1. $M$ is a **graph**, if for any members $a$ and $b$ of $D^M$, not $A^M(a, a)$, and if $A^M(a, b)$, then $A^M(b, a)$.

2. There is a **path** between $a$ and $b$ in $M$, if either $A^M(a, b)$ or there are finitely many members $x_1, x_2, \ldots, x_n$ of $D^M$ such that $A^M(a, x_1), A^M(x_1, x_2), \ldots, A^M(x_n, y)$.

3. $M$ is **connected**, if there is a path in $M$ between any two members of $D^M$.

\textsuperscript{54} I follow the usual practice of formulating the model theory in the set-theoretic metalanguage, but the reference to sets is not an essential part of formulating model theory. We can formulate it in higher-order plural languages. See Yi (2006, §6).

\textsuperscript{55} This is a variant of the theorem of undefinability of the class of even-numbered models (among finite models). The theorem can be proved using the method of the Ehrenfeucht-Fraïssé games. It can also be proved using the compactness of elementary logic. See, e.g., Väänänen (1994), who gives various proofs of the theorem.
Then the following is a theorem of finite model theory:\textsuperscript{56}

\begin{itemize}
  \item[(B*)] There is no sentence $\phi$ of $\mathcal{L}_d$ such that $\phi$ holds in a finite model $M$ of $\mathcal{L}_d$ if and only if $M$ is a connected graph.
\end{itemize}

Now, suppose that an elementary language sentence agrees with (3) on all finite models. (We may assume that such a sentence is in $\mathcal{L}_d$.) Then there is a sentence in $\mathcal{L}_d$ that agrees with the following on all finite models:

\begin{itemize}
  \item[(3a)] There is something such that there are some things that are not identical with it that admire only one another.
\end{itemize}

So let $\phi$ be such sentence. And let $\phi^*$ be

\[ [\forall x \sim A(x, x) \land \forall x \forall y(A(x, y) \lor A(y, x)) \land \phi]. \]

Then $\phi^*$ holds in a finite model $M$ of $\mathcal{L}_d$ if and only if $M$ is a connected graph. This contradicts (B*). So no elementary language sentence agrees with (3) on all finite models.

Department of Philosophy

\textsuperscript{56} The theorem can be proved using the method of the Ehrenfeucht-Fraïssé games. See, e.g., Ebbinghaus & Flum (1999, 22f) or Väänänen (1999, 9) for a proof.