Given a polygon with its vertices on lattice points, Pick’s Theorem gives us a simple formula for the area in terms of the number of lattice points in its interior and the number of lattice points on its boundary.

Our objective is to relate the area of a polygon on a lattice to the number of lattice points in its interior.

We consider polygons whose edges are midway between lattice points. For certain such polygons, called unimodular, we give a formula for the area in terms of the number of lattice points in its interior and the number of vertices.

**Pick’s Theorem**

Given a polygon with its vertices on lattice points, the area of the polygon is

\[ A = I + \frac{B}{2} - 1 \]

where \( I \) is the number of lattice points in its interior and \( B \) is the number of lattice points on its boundary.

For example, the pentagon in Figure 1 has an area of

\[ A = 7 + \frac{8}{2} - 1 = 10 \]

**Unimodular Polygons**

A polygon is called unimodular when the matrix of primitive edge vectors has determinant \( \pm 1 \) for each of its vertices. The triangle in Figure 2 is an example of a polygon that is unimodular.

We can see that the absolute value of the determinant of the matrix of primitive edge vectors is equal to 1 for each of the vertices:

\[
\begin{vmatrix}
0 & -1 \\
1 & -1
\end{vmatrix} = 1,
\]

\[
\begin{vmatrix}
0 & 1 \\
1 & 0
\end{vmatrix} = 1,
\]

\[
\begin{vmatrix}
-1 & 0 \\
-1 & 1
\end{vmatrix} = 1
\]

**Midway Lattice Polygons**

We consider polygons whose edges are midway between lattice points. For example, the hexagon in Figure 3 has edges midway between lattice points.

**Conjecture**

Given a unimodular polygon whose edges are midway between lattice points, the area of the polygon is

\[ A = I - \frac{V}{8} + \frac{1}{2} \]

where \( I \) is the number of lattice points in its interior and \( V \) is the number of vertices.

Here are some examples:

The area of the rectangle in Figure 4 is

\[ A = 8 - \frac{4}{8} + \frac{1}{2} = 8 \]

The area of the octagon in Figure 5 is

\[ A = 21 - \frac{8}{8} + \frac{1}{2} = \frac{41}{2} \]

**Discussion and Conclusion**

The next step in our project is to consider polygons whose edges are midway between lattice points that are not necessarily unimodular.

The two kites in Figure 6 are non-unimodular. They have the same area and the same primitive edge vectors, but the number of lattice points in their interior are different. Kite \( A \) has two lattice points in its interior while kite \( B \) has no lattice points in its interior. A similar example of this can be seen in Figure 7.

From these examples we conclude that an analogous formula in the non-unimodular case cannot only involve the primitive edge vectors and the number of lattice points in the interior; it will need to involve new ingredients.

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