LAGRANGIAN CALCULUS FOR NONSYMMETRIC DIFFUSION OPERATORS

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Abstract. We characterize lower bounds for the Bakry-Emery Ricci tensor of nonsymmetric diffusion operators by convexity of entropy and line integrals on L^2-Wasserstein space, and define a curvature-dimension condition for general metric measure spaces together with a square integrable 1-form in the sense of [Giga]. This extends the Lott-Sturm-Villani approach for lower Ricci curvature bounds of metric measure spaces. In generalized smooth context, consequences are new Bishop-Gromov estimates, pre-compactness under measured Gromov-Hausdorff convergence, and a Bonnet-Myers theorem that generalizes previous results by Kuwada [Kuw13]. We show that N-warped products together with lifted vector fields satisfy the curvature-dimension condition. For smooth Riemannian manifolds we derive an evolution variational inequality and contraction estimates for the dual semigroup of nonsymmetric diffusion operators. Another theorem of Kuwada [Kuw10, Kuw15] yields Bakry-Emery gradient estimates.

Contents

1. Introduction

In this article we present a Lagrangian approach for studying possibly nonsymmetric diffusion operators. For instance, we consider operators of the form $L = \Delta + \alpha$ where Δ is the Laplace-Beltrami operator of a compact smooth Riemannian manifold (M, g_M) and α is a smooth 1-form on M. Then the Bakry-Emery N-Ricci tensor associated to L is defined by

$$\mathrm{ric}^N_{M,\alpha} = \mathrm{ric}_M - \nabla^s \alpha - \tfrac{1}{N} \alpha \otimes \alpha$$

for $N \in (0, \infty]$ where $2\nabla^s \alpha(v, w) = \nabla_v \alpha + \nabla_w \alpha$ denotes the symmetric derivation of α with respect to the Levi-Civita connection ∇ of M. Note $\mathrm{ric}^N_{M,\alpha}$ is also meaningful for $N \leq 0$ but we will not consider these cases.

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If $\alpha = -df$, L is the diffusion operator of the canonical symmetric Dirichlet form associated to the smooth metric measure space (M, g_M, m_M) with $\mathrm{m}_M = e^{-f}\,\mathrm{vol}_M$, and $\mathrm{ric}^N_{M,-df}$ is understood as the Ricci curvature of (M, g_M, m_M). In celebrated articles by Lott, Sturm and Villani [LV09, Stu06a, Stu06b] - built on previous results in [OV00, CEMS01, vRS05] - a definition of lower Ricci curvature bounds for general metric measure spaces in terms of convexity properties of entropy functionals on the L^2-Wasserstein space was introduced. In smooth context these definitions are equivalent to lower bounds for the Bakry-Emery tensor provided α is exact.

For diffusion operators where α is not necessarily exact such a geometric picture was missing. Though the operator L yields a bilinear form, in general this form is not symmetric and therefore cannot arise as Dirichlet form of a metric measure space. Nevertheless, there are numerous results dealing with probabilistic, analytic and geometric properties of L under lower bounds on $\mathrm{ric}^N_{M,\alpha}$. The results are very similar to properties that one derives for symmetric operators with lower bounded Ricci curvature, e.g. [Kuw13, Kuw15, Wan11].

In this article we derive a geometric picture associated to the diffusion operator $L = \Delta + \alpha$ for general 1-forms α in the spirit of the work by Lott, Sturm and Villani. We characterize lower bounds on $\mathrm{ric}^N_{M,\alpha}$ in terms of convexity for line integrals along L^2-Wasserstein geodesics. Moreover, for generalized smooth metric measures spaces (Definition 2.1) we impose the following definition. For simplicity, in this introduction we assume $N = \infty$. We will say $(X, \mathrm{d}_X, \mathrm{m}_X)$ together with a 1-form α satisfies the curvature-dimension condition $CD(K, \infty)$ if and only if for every pair $\mu_0, \mu_1 \in \mathcal{P}^2(\mathrm{m}_X)$ there exists an L^2-Wasserstein geodesic Π such that

$$\begin{aligned}\mathrm{Ent}(\mu_t) - \alpha_t(\Pi) \leq (1-t)\,\mathrm{Ent}(\mu_0) + t\,[\mathrm{Ent}(\mu_1) - \alpha_1(\Pi)] \\ - \frac{1}{2}Kt(1-t)KW_2(\mu_0,\mu_1)^2,\end{aligned}$$

where $\alpha_t(\Pi) = \int\int_0^t \alpha(\dot\gamma(s))ds d\Pi(\gamma)$ and $\int_0^t \alpha(\dot\gamma(s))ds$ denotes the line integral of α along γ. For the case $N < \infty$ corresponding definitions are made in Definition 3.1 and Definition 3.2. In particular, we emphasize that Definition 3.2 is also meaningful in the class of general metric measure spaces together with L^2-integrable 1-forms in the sense of [Giga]. However, in this article we will only study the generalized smooth case.

We prove several geometric consequences: Generalized Bishop-Gromov estimates, pre-compactness under Gromov-Hausdorff convergence, and a generalized Bonnet-Myers Theorem. The latter generalizes a result of Kuwada in [Kuw13] - even for smooth ingredients. Then, we show that the condition $CD(K, N)$ is stable under N-warped product constructions. This also includes so-called euclidean N-cones and N-suspensions.

In the last section we introduce the notion of EVI_K-flows that arise naturally on generalized smooth metric measure spaces together with a 1-form satisfying a curvature-dimension condition. More precisely, if P_t is the semigroup associated to the operator $L = \Delta + \alpha$ on a smooth Riemannian manifold M such that $(M, g_M, \mathrm{vol}_M, \alpha)$ satisfies $CD(K, \infty)$, and if $\mathcal{H}_t$ is the dual flow acting on probability measures, then it is an absolutely continuous curve in L^2-Wasserstein space and for any probability measure μ, $\mathcal{H}_t\mu$ satisfies the following EVI-inequality

$$\frac{1}{2}\frac{d}{ds}W_2^2(\mathcal{H}_s\mu,\nu) + \frac{K}{2}W_2^2(\mathcal{H}_s\mu,\nu) \leq \int_0^1\int \alpha(\dot\gamma)d\Pi^s(\gamma)dt + \mathrm{Ent}(\nu) - \mathrm{Ent}(\mathcal{H}_s(\mu)$$

provided standard regularity assumptions for the corresponding heat kernel (Proposition 7.2). ν is any absolutely continuous probability measure, and Π^s is the L^2-Wasserstein geodesic between $\mathcal{H}_s\mu$ and ν. The EVI-inequality yields a contraction estimate for $\mathcal{H}_t$ (Corollary 7.5) that is again equivalent to the *so-called Bakry-Emery condition* $BE(K,\infty)$ for P_t:

$$|\nabla P_t f|^2 \leq e^{-2Kt} P_t |\nabla f|^2, \quad f \in W^{1,2}(M)$$

by [Kuw10]. Therefore, we obtain the following theorem.

Theorem 1.1. *Let (M, g_M) be a compact smooth Riemannian manifold, and let α be a smooth 1-form. We denote with $(M, \mathrm{d}_M, \mathrm{vol}_M)$ the corresponding metric measure space, and let P_t and $\mathcal{H}_t$ be as in the Section 7. Then, the following statements are equivalent.*

(i) $\mathrm{ric}^{\infty}_{M,\alpha} \geq K$,
(ii) $(M, \mathrm{d}_M, \mathrm{vol}_M, \alpha)$ *satisfies the condition* $CD(K,\infty)$,
(iii) *For every* $\mu \in \mathcal{P}^2(X)$ $\mathcal{H}_t\mu$ *is an* $EVI_{K,\infty}$*-flow curve starting in* μ,
(iv) $\mathcal{H}_t$ *satisfies the contraction estimate in Corollary 7.5,*
(v) P_t *satisfies the condition* $BE(K,\infty)$.

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2. Preliminaries

Metric measure spaces. Let (X, d_X) be a complete, separable metric space, and let m_X be a locally finite Borel measure. We call $(X, \mathrm{d}_X, \mathrm{m}_X)$ a metric measure space. The case $\mathrm{m}_X(X) = 0$ is excluded. The space of constant speed geodesics $\gamma : [0,1] \to X$ is denoted with $\mathcal{G}(X)$, and equipped with the topology of uniform convergence. $e_t : \gamma \mapsto \gamma(t)$ denotes the evaluation map at time t.

The L^2-Wasserstein space of probability measures with finite second moment is denoted with $\mathcal{P}^2(X)$, and W_2 is the L^2-Wasserstein distance. $\mathcal{P}^2_c(X)$ and $\mathcal{P}^2(\mathrm{m}_X)$ denote the subset of compactly supported probability measures and the family of m_X-absolutely continuous probability measures, respectively. A coupling or plan between probability measures μ_0 and μ_1 is a probability measure $\pi \in \mathcal{P}(X^2)$ such that $(p_i)_\star \pi = \mu_i$ where $(p_i)_{i=0,1}$ are the projection maps. A coupling π is optimal if $\int_{X^2} \mathrm{d}(x,y)^2 d\pi(x,y) = W_2(\mu_0,\mu_1)^2$. Optimal couplings exist, and if an optimal coupling π is induced by a map $T : Y \to X$ via $(\mathrm{id}_X, T)_\star \mu_0 = \pi$ where Y is a measurable subset of X, we say T is an optimal map.

A probability measure $\Pi \in \mathcal{P}(\mathcal{G}(X))$ is called an optimal dynamical coupling if $(e_0, e_1)_\star \Pi$ is an optimal coupling between its marginal distributions. Let $(\mu_t)_{t\in[0,1]}$ be an L^2-Wasserstein geodesic in $\mathcal{P}^2(X)$. We say an optimal dynamical coupling Π is a lift of μ_t if $(e_t)_\star \Pi = \mu_t$ for every $t \in [0,1]$. If Π is the lift of an L^2-Wasserstein geodesic μ_t, we call Π itself an L^2-Wasserstein geodesic. We say Π has bounded compression if there exists a constant $C := C(\Pi)$ such that $(e_t)_\star \Pi \leq C(\Pi)\, \mathrm{m}_X$ for every $t \in [0,1]$.

We say that a metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ is *essentially non-branching* if for any optimal dynamical coupling Π with bounded compression there exists $A \subset \mathcal{G}(X)$ such that $\Pi(A) = 1$ and for all $\gamma, \gamma' \in A$ we have that $\gamma(t) = \gamma'(t)$ for all $t \in [0, \epsilon]$ and for some $\epsilon > 0$ implies $\gamma = \gamma'$. For instance a metric measure space satisfying a Riemannian curvature-dimension condition $RCD(K, N)$ in the sense of [EKS15, Gigb] is essentially non-branching [RS14].

If we assume that for $\mathrm{m}_X \otimes \mathrm{m}_X$-almost every pair $(x, y) \in X^2$ there exists a unique geodesic $\gamma_{x,y} \in \mathcal{G}(X)$ between x and y then by measurable selection there exists a measurable map $\Psi : X^2 \to \mathcal{G}(X)$ with $\Psi(x, y) = \gamma_{x,y}$. For $\mu \in \mathcal{P}^2(X)$ and $A \subset X$ Borel with $\mathrm{m}_X(A) > 0$ we set $\mathcal{M}_{\mu,A} = \Psi_\star(\mu \otimes \mathrm{m}_X|_A)$, and for $x_0 \in X$ we set $\mathcal{M}_{\delta_{x_0},A} = \mathcal{M}_{x_0,A}$. In this case $\Pi_{x_0,A} := \mathrm{m}_X(A)^{-1}\mathcal{M}_{x_0,A}$ is the unique optimal dynamical plan between δ_{x_0} and $\mathrm{m}_X(A)^{-1}\,\mathrm{m}_X|_A$. Again, the family of RCD-spaces is a class that satisfies this property [GRS].

Definition 2.1 (Generalized smooth metric measure spaces)**.** We say $(X, \mathrm{d}_X, \mathrm{m}_X)$ is a *generalized smooth metric measure space* if there exists an open smooth manifold $M_X = M$, and a Riemannian metric g_M on M with induced distance function d_M such that the metric completion of the metric space (M, d_M) is isometric to (X, d_X), and for any optimal dynamical plan $\Pi \in \mathcal{P}(\mathcal{G}(X))$ such that $(e_t)_\star\Pi = \mu_t$ is a geodesic and $\mu_0 \in \mathcal{P}^2(\mathrm{m}_X)$ we have that

$$\Pi(S_\Pi) = 0 \quad \text{where} \quad S_\Pi := \{\gamma \in \mathcal{G}(X) : \exists t \in (0,1) \text{ s.t. } \gamma(t) \in X \backslash M\}\,. \tag{1}$$

In particular, $\mu_t(X\backslash M) = \Pi(e_t^{-1}(X\backslash M)) \leq \Pi(S_\Pi) = 0$, and if we choose the constant geodesic Π with $(e_t)_\star\Pi = \mathrm{m}_X(K)^{-1}\,\mathrm{m}_X|_K$ for all $t \in [0, 1]$ where $K \subset X$ is any measurable set of finite m_X-measure, one gets that $K \cap X\backslash M$ is of m_X-measure 0. We call M the set of regular points in X.

Remark 2.2. The condition (1) yields that $(X, \mathrm{d}_X, \mathrm{m}_X)$ is essentially non-branching and that for $\mathrm{m}_X \otimes \mathrm{m}_X$-almost every pair $(x, y) \in X^2$ there exists a unique geodesic $\gamma_{x,y} \in \mathcal{G}(X)$ between x and y. Moreover, for each pair $\mu_0, \mu_1 \in \mathcal{P}^2_c(\mathrm{m}_X)$ there is a unique dynamical optimal coupling Π such that $\Pi(\mathcal{G}(M)) = 1$ where $\mathcal{G}(M)$ is the space of geodesic in M, $(e_t)_\star\Pi \in \mathcal{P}^2(\mathrm{m}_X)$, and Π is induced by a map. To see this note that $\mu_i(M) = 1, i = 0, 1$ and transport geodesics are contained in M. Then, since one can choose an exhaustion of M by compact sets, we can assume that μ_0 and μ_1 are compactly supported in M. Then, the claim follows from statements in [CEMS01] and since geodesics are unique.

Examples of generalized smooth metric measure spaces in the sense of the previous definition are Riemannian manifolds with boundary that are geodesically convex, cones and suspensions over spaces with diameter less than π [BS14] and warped products under natural assumptions [Ket13].

1-forms and vector fields. Assume $(X, \mathrm{d}_X, \mathrm{m}_X)$ is a generalized smooth metric measure space. A 1-form α is a measurable map $\alpha : X \to T^*M$ with $\alpha(x) = T^*_xM$. We say $\alpha \in L^p_{loc}(\mathrm{m}_X, TM^*)$ for $p \in [1, \infty)$ if $\|\alpha\|^p_{L^p(\mathrm{m}_X,K)} = \int_{K\cap M} |\alpha|^p_M d\,\mathrm{m}_X$ is finite where $|\alpha|^2_M = g^*_M(\alpha, \alpha)$ and $K \subset X$ compact.

Similar, we can consider measurable and L^p-integrable vector fields on X. Note that a vector field Z on M yields a 1-form α via $\alpha = \langle Z, \cdot\rangle$. In the context of generalized smooth metric measures space this is the natural isomorphism between

vector fields and 1-forms, and we will often switch between these viewpoints. If Π is an optimal dynamical coupling with bounded compression, the line integral

$$\alpha_t(\gamma) := \int_0^t \alpha(\dot\gamma)(\tau)d\tau$$

exists Π-almost surely, and it does not depend on the parametrization of γ up to changes of orientation. Moreover, for any L^2-Wasserstein geodesic Π with bounded compression, we set $\alpha_t(\Pi) = \int \alpha_t(\gamma)d\Pi(\gamma)$.

The case of arbirtrary metric measure spaces. Let $\phi : X \to \mathbb{R} \cup \{\pm\infty\}$ be any function. The Hopf-Lax semigroup $Q_t : X \to \mathbb{R} \cup \{-\infty\}$ is defined by $Q_t\phi(x) = \inf_y \frac{1}{2t}\,\mathrm{d}(x,y)^2 + \phi(y)$. $Q_1(-\phi)$ is the c-transform of ϕ. We say a function ϕ is c-concave if there exist $v : X \to \mathbb{R}\cup\{\pm\infty\}$ such that $\phi = Q_1 v$. If X is compact, by Kantorovich duality for any pair $\mu_0, \mu_1 \in \mathcal{P}^2(\mathrm{m}_X)$ with bounded densities there exist a Lipschitz function ϕ such that

$$W_2(\mu_0,\mu_t)^2 = \int Q_t\phi d\mu_1 - \int \phi d\mu_0 = \int_0^t \int \frac{d}{ds} Q_s\phi\big|_{s=t}\rho_t d\,\mathrm{m}_X\, dt \tag{2}$$

for any geodesic $t \mapsto \mu_t$ with bounded compression. For instance, see [GH].

If we follow the approach of Gigli in [Giga], there is also a well-defined notion of $L^p(\mathrm{m}_X)$-integrable 1-form α for general metric measure spaces $(X, \mathrm{d}_X, \mathrm{m}_X)$, and one can define the dual coupling $\alpha(\nabla f) : X \to \mathbb{R}$ as measurable function on X where f is a Sobolev function. Note that in this context ∇f does not necessarily exist. The notion of line integral along a geodesic is more subtle, but if we consider a L^2-Wasserstein geodesic $\Pi \in \mathcal{P}(\mathcal{G}(X))$ that has bounded compression, then we can define $\bar\alpha_t(\Pi) := -\int_0^t \int \alpha(\nabla Q_s\phi)\rho_s d\,\mathrm{m}_X\, ds$ where $Q_t\phi$ is a Kantorovich potential for Π, and ρ_t is the density of $(e_t)_\star\Pi$. Since ρ_t and $\nabla Q_t\phi$ are bounded, $\bar\alpha_t(\Pi)$ is well-defined if $|\alpha|$ is m_X-integrable. Note that in a smooth context $\dot\gamma(t) = -\nabla Q_t\phi|_{\gamma(t)}$, and therefore in smooth context we have

$$\begin{aligned}\bar\alpha_t(\Pi) &= \int_0^t \int \alpha(\nabla Q_s\phi|_x)\rho_s(x)d\,\mathrm{m}_X(\{x\})ds = \int_0^t \int \alpha(\nabla Q_s\phi|_{\gamma(s)})d\Pi(\gamma)ds \\ &= \int\int_0^t \alpha(\nabla Q_s\phi|_{\gamma(s)})dsd\Pi(\gamma) = \int \alpha_t(\gamma)d\Pi(\gamma) = \alpha_t(\Pi).\end{aligned}$$

In the following we just write $\alpha_t(\Pi)$ for $\bar\alpha_t(\Pi)$. Also note, that in general there is no identification between 1-forms and vector fields.

Entropy functionals. For $\mu \in \mathcal{P}^2(X)$ we define the Boltzmann-Shanon entropy by

$$\mathrm{Ent}(\mu) := \int \log\rho d\mu \quad \text{if} \quad \mu = \rho\,\mathrm{m}_X \ \text{ and } (\rho\log\rho)_+ \text{ is } \mathrm{m}_X\text{-integrable},$$

and $\mathrm{Ent}(\mu) = +\infty$ otherwise. Given a number $N \geq 1$, we define the N-Rényi entropy functional $S_N : \mathcal{P}^2(X) \to (-\infty, 0]$ with respect to m_X by

$$S_N(\mu) := -\int \rho^{1-\frac{1}{N}}(x)d\,\mathrm{m}_X$$

where ρ denotes the density of the absolutely continuous part in the Lebesgue decomposition of μ. In the case $N = 1$ the 1-Rényi entropy is $-\mathrm{m}_X(\operatorname{supp}\rho)$. If m_X is finite, then

$$-\mathrm{m}_X(X)^{\frac{1}{N}} \leq S_N(\cdot) \leq 0$$

and $\mathrm{Ent}(\mu) = \lim_{N\to\infty} N(1+S_N(\mu))$. Moreover, if m_X is finite and $N>1$, then S_N is lower semi-continuous. If m_X is σ-finite one has to assume an exponential growth condition [AGMR15] to guarantee lower semi continuity.

If there is a 1-form α, we also define $S^\alpha_{N,t} : \mathcal{P}(\mathcal{G}(X)) \to (-\infty, 0]$ by

$$S^\alpha_{N,t}(\Pi) := -\int \rho_t^{-\frac{1}{N}}(\gamma_t) e^{\frac{1}{N}\alpha_t(\gamma)} d\Pi(\gamma) \quad \text{if } (e_t)_\star \Pi = \rho_t \, \mathrm{m}_X$$

and 0 otherwise. If $\alpha = 0$, then $S^\alpha_{N,t}(\Pi) = S_N((e_t)_\star \Pi)$.

Distortion coefficients. For two numbers $K \in \mathbb{R}$ and $N \geq 1$ we define

$$(t,\theta) \in [0,1]\times(0,\infty) \mapsto \sigma^{(t)}_{K,N}(\theta) = \begin{cases} \frac{\sin_{K/N}(t\theta)}{\sin_{K/N}(\theta)} & \text{if } \sin_{K/N}(x) > 0 \text{ for } x \in (0,\theta], \\ \infty & \text{otherwise.} \end{cases}$$

$\sin_{K/N}$ is the solution of the initial value problem

$$u'' + \tfrac{K}{N} u = 0, \quad u(0) = 0 \ \& \ u'(0) = 1.$$

The modified distortion coefficients for number $K \in \mathbb{R}$ and $N > 1$ are given by

$$(t,\theta) \in [0,1]\times(0,\infty) \mapsto \tau^{(t)}_{K,N}(\theta) = \begin{cases} \theta \cdot \infty & \text{if } K > 0 \text{ and } N = 1, \\ t^{\frac{1}{N}} \left[\sigma^{(t)}_{K/(N-1)}(\theta)\right]^{1-\frac{1}{N}} & \text{otherwise.} \end{cases}$$

3. Curvature-dimension condition for nonsymmetric diffusions

Definition 3.1. Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a generalized smooth metric measure space, and let α be an L^2-integrable 1-form. We say $(X, \mathrm{d}_X, \mathrm{m}_X, \alpha)$ satisfies the *curvature-dimension condition* $CD(K,N)$ for $K \in \mathbb{R}$ and $N \geq 1$ if and only if for each pair $\mu_0, \mu_1 \in \mathcal{P}^2_c(X, \mathrm{m}_X)$ there exists a dynamical optimal plan Π with

$$S^\alpha_{N,t}(\Pi) \leq -\int \left[\tau^{(1-t)}_{K,N}(|\dot\gamma|)\rho_0(\gamma_0)^{-\frac{1}{N}} + \tau^{(t)}_{K,N}(|\dot\gamma|) e^{\frac{1}{N}\alpha_t(\gamma)} \rho_1(\gamma_1)^{-\frac{1}{N}}\right] d\Pi(\gamma)$$

where $\alpha_t(\gamma) = \int_0^t \alpha(\dot\gamma)(\tau) d\tau$.

If we replace $\tau^{(t)}_{K,N}(\theta)$ by $\sigma^{(t)}_{K,N}(\theta)$ in the previous definition we say $(X, \mathrm{d}_X, \mathrm{m}_X, \alpha)$ satisfies the *reduced curvature-dimension condition* $CD^*(K,N)$.

Definition 3.2. Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space, and let α be an L^2-integrable 1-form in the sense of [Giga].

We say $(X, \mathrm{d}_X, \mathrm{m}_X, \alpha)$ satisfies the *curvature-dimension condition* $CD(K,\infty)$ if and only if for each pair $\mu_0, \mu_1 \in \mathcal{P}^2(\mathrm{m}_X)$ with bounded densities there exists a geodesic Π with bounded compression and a potential ϕ as in (2) such that

$$\begin{aligned} \mathrm{Ent}(\mu_t) - \alpha_t(\Pi) \leq (1-t)\,\mathrm{Ent}(\mu_0) + t\,[\mathrm{Ent}(\mu_1) - \alpha_1(\Pi)] \\ - \frac{1}{2} K t(1-t) K W_2(\mu_0,\mu_1)^2, \end{aligned}$$

where $\alpha_t(\Pi) = \int_0^t \int \alpha(\nabla Q_\tau \phi)\rho_\tau d\,\mathrm{m}_X \, d\tau$ and $\mu_t = (e_t)_\star \Pi$. Equivalently, the map $t \mapsto \mathrm{Ent}(\mu_t) - \alpha_t(\Pi)$ is K-convex.

$(X, \mathrm{d}_X, \mathrm{m}_X, \alpha)$ satisfies the *entropic curvature-dimension condition* $CD^e(K,N)$ if and only if for each pair $\mu_0, \mu_1 \in \mathcal{P}^2(\mathrm{m}_X)$ with bounded densities there exists a geodesic Π with bounded compression and a potential ϕ as in (2) such that

$$U_N(\mu_t) e^{\frac{1}{N}\alpha_t(\Pi)} \leq \sigma^{(1-t)}_{K,N}(W_2(\mu_0,\mu_1)) U_N(\mu_0) + \sigma^{(t)}_{K,N}(W_2(\mu_0,\mu_1)) e^{\frac{1}{N}\alpha_1(\Pi)} U_N(\mu_1)$$

where $U_N(\mu) = e^{-\frac{1}{N}\operatorname{Ent}(\mu_t)}$ and $\mu_t = (e_t)_\star \Pi$. That is the map $t \mapsto \operatorname{Ent}(\mu_t) - \alpha_t(\Pi)$ is (K, N)-convex in the sense of [EKS15].

Remark 3.3. If $(X, d, \mathrm{m}, 0)$ satisfies a curvature-dimension condition $CD(K, N)$, $CD^*(K, N)$ or $CD^e(K, N)$ as in the previous definitions, (X, d, m) satisfies a condition $CD(K, N)$, $CD^*(K, N)$ or $CD^e(K, N)$ as introduced in [LV09, Stu06a, Stu06b], [BS10] or [EKS15] respectively.

Remark 3.4. It is easy to prove that

(i) $CD(K, N) \implies CD^*(K, N)$,
(ii) $CD^*(K, N), CD^e(K, N) \implies CD^*(K', N'), CD^e(K', N')$ for $K' \leq K$ and $N' \geq N$,
(iii) If m_X is finite, then $CD^*(K, N), CD^e(K, N) \implies CD(K, \infty)$.

For instance, compare with [BS10, Proposition 2.5], [Stu06b, Proposition 1.6] and [EKS15, Lemma 3.2]

Definition 3.2 makes sense for any metric measure space. But for simplicity, for the rest of the article we always assume that $(X, \mathrm{d}_X, \mathrm{m}_X)$ *is a generalized smooth metric measure space.*

Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ and $(X', \mathrm{d}_{X'}, \mathrm{m}_{X'})$ be generalized smooth metric measure spaces. A map $I : \operatorname{supp} \mathrm{m}_{X'} \to X$ is a smooth metric measure space isomorphism if I is a metric measure space isomorphism, and if I is a diffeomophism between the subsets of regular points M' and M.

Proposition 3.5. *Let* $(X, \mathrm{d}_X, \mathrm{m}_X) = \mathrm{X}$ *be a generalized smooth metric measure space, and let* α *be an* L^2*-intergrable* 1*-form. Assume* (X, α) *satisfies the condition* $CD(K, N)$*. Then the following properties hold.*

(i) *For* $\eta, \beta > 0$ *define the generalized smooth metric measures space* $(X, \eta\, \mathrm{d}_X, \beta\, \mathrm{m}_X) =: \mathrm{X}'$*. Then* (X', α) *satisfies* $CD(\eta^{-2}K, N)$*.*
(ii) *For a convex subset* $X' \subset X$ *define the generalized smooth metric measure space* $(X', \mathrm{d}\,|_{X' \times X'}, \mathrm{m}\,|_{X'}) =: \mathrm{X}'$*. Then* $(\mathrm{X}', \alpha|_{X'})$ *satisfies* $CD(K, N)$*.*
(iii) *Let* X' *be a generalized smooth metric measure space, and let* $I : X' \to X$ *be a smooth metric measure space isomorphism. Then* $(\mathrm{X}', I^\star\alpha)$ *satisfies the condition* $CD(K, N)$*.*

Proof. We check (iii). We define $\alpha' = I^\star\alpha$ on $I^{-1}(M)$. If γ is a geodesic in X, then $I^{-1} \circ \gamma = \gamma'$ is a geodesic in X'. The line integral of α' along γ' is

$$\int_0^1 \alpha'(\dot{\gamma}')dt = \int_0^1 I^\star\alpha(DI^{-1}|_{\gamma(t)}\dot{\gamma})dt = \int_0^1 \alpha(\dot{\gamma})dt.$$

Then, the statement follows like similar results for metric measure spaces that satisfy a curvature-dimension condition (for instance see [Stu06a, Proposition 4.12]). □

Theorem 3.6. *Let* $(X, \mathrm{d}_X, \mathrm{m}_X) = \mathrm{X}$ *be a generalized smooth metric measure space,* α *an* L^2*-integrable* 1*-form, and* $K \in \mathbb{R}$ *and* $N > 0$*. Then the following statements are equivalent:*

(i) (X, α) *satisfies* $CD^*(K, N)$*.*

(ii) *For each pair* $\mu_0, \mu_1 \in \mathcal{P}_c^2(\mathrm{m}_X)$ *there exists an optimal dynamical plan* Π *with* $(e_t)_\star \Pi = \mu_t \in \mathcal{P}^2(\mathrm{m}_X)$ *such that*

$$\left[\rho_t(\gamma_t) e^{-\alpha_t(\gamma)}\right]^{-\frac{1}{N}} \geq \sigma_{K,N}^{(1-t)}(|\dot\gamma|)\rho_0(\gamma_0)^{-\frac{1}{N}} + \sigma_{K,N}^{(1-t)}(|\dot\gamma|)\left[e^{-\alpha_t(\gamma)}\rho_1(\gamma_1)\right]^{-\frac{1}{N}} \tag{3}$$

for $t \in [0,1]$ *and* Π*-a.e.* $\gamma \in \mathcal{G}(X)$*.* ρ_t *is the density of* μ_t *w.r.t.* m_X*.*

(iii) (X, α) *satisfies* $CD^e(K,N)$*.*

Moreover, the condition $CD(K,N)$ *is equivalent with (ii) if the coefficients* $\sigma_{K,N}^{(t)}(\theta)$ *are replaced by the coefficients* $\tau_{K,N}^{(t)}(\theta)$*.*

Proof. First, we observe that in the context of generalized smooth metric measure spaces up to a set measure zero optimal couplings between m_X-absolutly continuous measures $\mu_0, \mu_1 \in \mathcal{P}_c(\mathrm{m}_X)$ are unique (also compare with the remark after Definition 2.1).

"(i)⇒(ii)": Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathrm{m}_X)$ be with bounded support, and let $\pi \in \mathcal{P}(\mathcal{G}(X))$ be the optimal coupling between μ_0 and μ_1. Let $\{M_n\}_{n\in\mathbb{N}}$ be an $\cap$-stable generator of the Borel σ-field of (X, d_X) . For each n we define a disjoint covering of X of 2^n sets by $L_I = \bigcap_{i\in I} M_i \cap \bigcap_{i\in I^c} M_i^c$ where $I \subset \{1,\dots,n\}$ and $I^c = \{1,\dots,n\}\setminus I$.

We define $B^{I,J} = L_I \times L_J$ and set $\pi^{I,J} := \alpha_{I,J}^{-1}\pi|_{B^{I,J}}$ if $\alpha_{I,J} := \pi(B^{I,J}) > 0$. Then we consider the marginal measures $\mu_0^{I,J} = (e_0)_\star \pi^{I,J}$ and $\mu_1^{I,J} = (e_1)_\star \pi^{I,J}$ that are m_X-absolutely continuous, and $\pi^{I,J}$ is the unique optimal coupling. Since geodesics are $\mathrm{m}_X \otimes \mathrm{m}_X$-almost surely unique, the dynamical optimal plan $\Pi^{I,J} = \Psi_\star \pi^{I,J}$ is the unique optimal dynamical coupling between its endpoints where $\Psi(x,y) = \gamma_{x,y} \in \mathcal{G}(X)$. Therefore, $\Pi^{I,J}$ satisfies the CD^*-inequality for every I and J. In particular, $(e_t)_\star \Pi^{I,J} = \rho_t^{I,J}\,\mathrm{m}_X$ is m_X-absolutely continuous. Then, we define a dynamical coupling between μ_0 and μ_1 by $\Pi^n := \sum_{I,J\subset\{1,\dots,n\}} \alpha_{I,J}\Pi^{I,J}$. Π^n is optimal since

$$\pi^n := (e_0,e_1)_\star \Pi_n = \sum_{I,J\subset\{1,\dots,n\}} \alpha_{I,J}(e_0,e_1)_\star \Pi^{I,J} = \sum_{I,J\subset\{1,\dots,n\}} \alpha_{I,J}\pi^{I,J} = \pi$$

is an optimal coupling. Therefore, we can apply Lemma 3.11 in [EKS15]: Since the measures $\mu_0^{I,J}$ for $I, J \subset \{1,\dots,2^n\}$ are mutually singular, $\mu_t^{I,J} = \rho_t^{I,J} d\,\mathrm{m}_X$ are mutually singular as well.

Now, for $t \in (0,1)$ we consider the measure $\mu_t^n = (e_t)_\star \Pi^n$. Since it decomposes into mutually singular, absolutely continuous measures $\mu_t^{I,J}$ with densities $\rho_t^{I,J}$, μ_t^n is absolutely continuous as well, and by mutual singularity of the measure $\mu_t^{I,J}$ its density is $\rho_t^n = \sum \alpha_{I,J}\rho_t^{I,J}$. Again, since geodesics are $\mathrm{m}_X \otimes \mathrm{m}_X$-almost surely unique we have that $\Pi := \Psi_\star\pi = \Psi_\star\pi^n = \Pi^n$, and $\rho_t\,\mathrm{m}_X = (e_t)_\star\Pi = (e_t)_\star\Pi^n = \rho_t^n\,\mathrm{m}_X$ for every $n \in \mathbb{N}$. From the CD^*-inequality for $\Pi^{I,J}$ we have

$$\begin{aligned}
&\int_{L_I\times L_J} \rho_t^{-\frac{1}{N}}(\gamma_{x,y}(t)) e^{-\frac{1}{N}\alpha_t(\gamma_{x,y})} d\pi(x,y)\\
&= \alpha_{I,J}^{1-\frac{1}{N}} \int (\rho_t^{I,J})^{-\frac{1}{N}}(\gamma_{x,y}(t)) e^{-\frac{1}{N}\alpha_t(\gamma_{x,y})} d\pi^{I,J}(x,y)\\
&\geq \alpha_{I,J}^{1-\frac{1}{N}} \int \sigma_{K,N}^{(1-t)}(|\dot\gamma_{x,y}|)(\rho_0^{I,J})^{\frac{-1}{N}}(x) + \sigma_{K,N}^{(t)}(|\dot\gamma_{x,y}|)(\rho_1^{I,J})^{\frac{-1}{N}}(y) e^{\frac{-1}{N}\alpha_t(\gamma_{x,y})} d\pi^{I,J}(x,y)\\
&= \int_{L_I\times L_J} \sigma_{K,N}^{(1-t)}(|\dot\gamma_{x,y}|)\rho_0^{-\frac{1}{N}}(x) + \sigma_{K,N}^{(t)}(|\dot\gamma_{x,y}|)\rho_1^{-\frac{1}{N}}(y) e^{-\frac{1}{N}\alpha_t(\gamma_{x,y})} d\pi(x,y).
\end{aligned}$$

This holds for every L_I and L_J. Since L_I and L_J are mutually disjoint, by summing up the previous inequality holds for M_i and M_j as well. Since the family $\{M_j\}$ is a $\cap$-stable generator for the σ-field, we have for π-almost every $(x,y) \in X \times X$

$$\rho_t^{-\frac{1}{N}}(\gamma_{x,y}(t))e^{-\frac{1}{N}\alpha_t(\gamma_{x,y})} \geq \sigma_{K,N}^{(1-t)}(|\dot{\gamma}_{x,y}|)\rho_0^{-\frac{1}{N}}(x) + \sigma_{K,N}^{(t)}(|\dot{\gamma}_{x,y}|)\rho_1^{-\frac{1}{N}}(y)e^{-\frac{1}{N}\alpha_t(\gamma_{x,y})}.$$

And since $\Pi = \Psi_\star \pi$, this is the claim.

For the following recall that in the context of generalized smooth metric measure spaces

$$\int_0^1 \int \alpha(\nabla Q_t \phi)\rho_t dt = \int \int_0^1 \alpha(\dot{\gamma}) dt d\Pi$$

where Π is an L^2-Wasserstein geodesic, ϕ is an Kantorovich potential, and $(e_t)_\star \Pi = \rho_t \, \mathrm{m}_X$ for every $t \in [0,1]$.

"(ii)$\Rightarrow$(iii)": Recall from [EKS15, Lemma 2.11] that $(x,y,\theta) \mapsto G(x,y,\theta) = \log(\sigma_{K,N}^{(t)}(\theta)e^x + \sigma_{K,N}^{(t)}(\theta)e^y)$ is convex. Then, apply log to (3) and use Jensen's inequality on the right hand side to obtain the condition $CD^e(K,N)$.

"(iii)$\Rightarrow$(ii)": This works like in "(i)$\Rightarrow$(ii)" where one has to use the convexity of $(x,y,\theta) \mapsto G(x,y,\theta)$ again. See also [EKS15, Lemma 2.11].

"(ii)$\Rightarrow$(i)": Integrate (3) w.r.t. the optimal dynamical plan Π.

The proof of the equivalence for the condition $CD(K,N)$ is similar. □

Remark 3.7. If we consider α such that $\alpha = -df$ on M for a smooth function $f : M \to \mathbb{R}$, then

$$\alpha_t(\gamma) = \int_0^t \alpha(\dot{\gamma})dt = f(\gamma(t)) - f(\gamma(0)) \;\; \& \;\; \alpha_t(\gamma) = f(\gamma(1)) - f(\gamma(0)),$$

and we can reformulate (3) as

$$\left[\rho_t(\gamma_t)e^{f(\gamma_t)}\right]^{-\frac{1}{N}} \geq \tau_{K,N}^{(1-t)}(|\dot{\gamma}|)\left[\rho_0(\gamma_0)e^{f(\gamma_0)}\right]^{-\frac{1}{N}} + \tau_{K,N}^{(1-t)}(|\dot{\gamma}|)\left[\rho_1(\gamma_1)e^{f(\gamma_1)}\right]^{-\frac{1}{N}}$$

for Π-a.e. γ. That is the condition $CD(K,N)$ in the sense of [Stu06b] for the metric measure space $(M, \mathrm{d}_X, e^{-f}\,\mathrm{vol}_X)$.

4. The Riemannian manifold situation

In this section we consider a vector field Z rather than a 1-form α. Recall that for smooth metric measure spaces we always can identifiy Z with a 1-form α.

Definition 4.1. Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a smooth metric measure space with $(X, \mathrm{d}_X) \simeq (M, \mathrm{d}_M)$ and $\mathrm{m}_X = \mathrm{vol}_M$. Let ∇ be the Levi-Civita connection of g_M and let $Z \in L^2(TM)$ be a smooth vector field. We define the Bakry-Emery N-Ricci tensor for $N \in (n, \infty]$ by

$$\mathrm{ric}_{M,Z}^N = \mathrm{ric}_M - \nabla^s Z - \frac{1}{N-n} Z \otimes Z$$

where $\nabla^s Z(v,w) = \frac{1}{2}\left(\langle \nabla_v Z, w\rangle + \langle v, \nabla_w Z\rangle\right)$ and $n = \dim_M$. For $N = n$ we define

$$\mathrm{ric}_{M,Z}^N(v) := \begin{cases} \mathrm{ric}_M(v) - \nabla^s Z(v) & \langle Z, v\rangle_F = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

For $1 \leq N < n$ we define $\mathrm{ric}_{M,Z}^N(v) := -\infty$ for all $v \neq 0$ and 0 otherwise.

Theorem 4.2. *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a smooth metric measure space with $(X, \mathrm{d}_X) \simeq (M, \mathrm{d}_M)$ and $\mathrm{m}_X = \mathrm{vol}_M$, $K \in \mathbb{R}$ and $N \in [1, \infty]$. Let Z be an L^2-integrable smooth vector field. Then $(X, \mathrm{d}_X, \mathrm{m}_X, Z)$ satisfies the condition $CD(K, N)$ if and only if*

$$\mathrm{ric}^N_{M,Z}(v) \geq K|v|^2. \tag{4}$$

Moreover, if N is finite, then $CD(K, N)$ and $CD^*(K, N)$ are equivalent.

Proof. **1.** Assume $\mathrm{ric}^N_{M,Z} \geq K g_M$. Let $N < \infty$. The case $N = \infty$ follows by obvious modifications. Consider $\mu_0, \mu_1 \in \mathcal{P}^2(M)$ that are compactly supported. Otherwise we can choose compact exhaustions of $M \times M$ and consider the restriction of optimal couplings to these sets. There exists a c-concave function ϕ such that $T_t(x) = \exp_x(-t\nabla\alpha_x)$ is the unique optimal map between μ_0 and $(T_t)_*\mu_0 = \mu_t$, and μ_t is the unique L^2-Wasserstein geodesic between μ_0 and μ_1 in $\mathcal{P}^2(M)$. μ_t is compactly supported and m_X-absolutely continuous [CEMS01].

The potential ϕ is semi-concave, and by the Bangert-Alexandrov theorem the Hessian $\nabla^2\phi(x)$ exists for m_X-almost every $x \in M$. Moreover, for m_X-almost every $x \in M$ the optimal map T_t admits a Jacobian $DT_t(x)$ for every $t \in [0, 1]$. $DT_t(x)$ is non-singular for every $t \in [0, 1]$, and the Monge-Ampere equation

$$\rho_0(x) = \det DT_t(x)\rho_t(T_t(x)) \tag{5}$$

holds m_X-almost everywhere.

2. We pick a point $x \in M$ where $DT_t(x)$ is non-singular and (5) holds. For an orthonormal basis $(e_i)_{i=1,\dots,n}$ of T_xM

$$t \mapsto V_i(t) = DT_t(x)e_i \in T_{\gamma_x(t)}M$$

is the Jacobi field along $\gamma_x(t) = \exp_x(-t\nabla\phi|_x) = T_t(x)$ with initial condition $V_i(0) = e_i$ and $V_i'(0) = -\nabla_{e_i}\nabla\phi|_x$. Therefore

$$V_i''(t) + R(V_i(t), \dot\gamma(t))\dot\gamma(t) = 0 \quad \text{for } i = 1, \dots, n.$$

We set $\mathcal{A}_t(x) = (V_1, \dots, V_n)$. Since $DT_t(x)$ is non-singular for every $t \in [0, 1]$, Riemannian Jacobi field calculus (for instance see [Stu06b, Proof of Theorem 1.7]) yields for $t \mapsto y_t = \log\det\mathcal{A}_t(x)$ the differential inequality

$$y_t'' \leq -\frac{1}{n}(y_t')^2 - \mathrm{ric}_X(\dot\gamma, \dot\gamma). \tag{6}$$

3. Consider the vector field Z, the geodesic $t \in [0, 1] \to \gamma_x(t)$, and the corresponding line integral $t \mapsto \phi^Z_t(\gamma_x) =: \alpha_t$. We set $\gamma := \gamma_x$ and compute

$$\phi_t'' = \langle \nabla_{\dot\gamma} Z|_{\gamma(t)}, \dot\gamma(t)\rangle + \langle Z|_{\gamma(t)}, \nabla_{\dot\gamma}\dot\gamma(t)\rangle = \langle \nabla_{\dot\gamma} Z|_{\gamma(t)}, \dot\gamma(t)\rangle = \nabla^s Z(\dot\gamma, \dot\gamma).$$

In addition with (6) and (4) this yields

$$\begin{aligned} y_t'' + \phi_t'' &\leq -\frac{1}{n}(y_t')^2 - \mathrm{ric}_X(\dot\gamma, \dot\gamma) + \nabla^s Z(\dot\gamma, \dot\gamma) \\ &\leq -K|\dot\gamma|^2 - \frac{1}{N-n} Z \otimes Z(\dot\gamma, \dot\gamma) - \frac{1}{n}(y_t')^2 \\ &\leq -K|\dot\gamma|^2 - \frac{1}{N}\left(y_t' + Z(\dot\gamma)(t)\right)^2. \end{aligned}$$

Hence

$$y_t'' + \phi_t'' + \frac{1}{N}\left(y_t' + \phi_t'\right)^2 + K|\dot\gamma|^2 \leq 0.$$

If we set $\mathcal{I}(t) = e^{y_t+\alpha_t}$, then we obtain

$$\frac{d^2}{dt^2}\mathcal{I}^{\frac{1}{N}} \leq -\frac{K|\dot\gamma|^2}{N}\mathcal{I}^{\frac{1}{N}},$$

or equivalently

$$\mathcal{I}_t^{\frac{1}{N}} \geq \sigma_{K,N}^{(1-t)}(|\dot\gamma|)\mathcal{I}_0^{\frac{1}{N}} + \sigma_{K,N}^{(t)}(|\dot\gamma|)\mathcal{I}_1^{\frac{1}{N}}. \tag{7}$$

Note the dependence on $x \in M$. (7) holds for m_X-a.e. $x \in M$.

4. By the same computation as in [Stu06b] we can improve (7) by taking out the direction of motion if $n \geq 2$. We know that

$$U'(t) + U^2(t) + R(t) = 0 \tag{8}$$

where $U(t) = \mathcal{A}'(t)\mathcal{A}(t)^{-1}$ and $R_{i,j}(t) = \langle R(V_i, \dot\gamma)\dot\gamma, V_j\rangle$. From Lemma 3.1 in [CEMS06] one sees that U is symmetric. R has the form

$$R(t) = \begin{pmatrix} 0 & 0 \\ 0 & \bar R(t) \end{pmatrix}$$

for an $(n-1)\times(n-1)$-matrix $\bar R$. Hence, if $U = (u_{i,j})_{i,j=1,\dots,n}$, we have

$$u_{11}' + \sum_{i=1}^n u_{1,i}^2 = 0. \tag{9}$$

Moreover, taking the trace in (8) yields

$$\operatorname{tr} U' + \operatorname{tr} U^2 + \mathrm{ric} = 0. \tag{10}$$

where $\mathrm{ric} = \mathrm{ric}(\dot\gamma)$. The Jacobi determinant $\mathcal{J}_t = \det \mathcal{A}_t$ satisfies $(\log \mathcal{J})' = \mathcal{J}'/\mathcal{J} = \operatorname{tr} U = u_{11} + \sum_{i=2}^n u_{ii}$. Therefore if we set $\lambda_t = \int_0^t u_{11}(s)ds$, we have

$$y_t = \log \mathcal{J}_t = \lambda_t + \int_0^t \Big[\sum_{i=2}^n u_{ii}(s)\Big]ds$$

and (10) becomes $y'' + \operatorname{tr} U^2 + \mathrm{ric} = 0$. λ_t describes volume distortion in direction of the transport geodesics. We remove this part and consider $\bar y_t := y_t - \lambda_t$. And if we set $\bar U = (u_{ij})_{i,j=2,\dots n}$, then $\bar y_t = \int_0^t \operatorname{tr}\bar U ds$. A computation yields

$$\begin{aligned}\bar y_t'' = y_t'' - \lambda_t'' &= -\operatorname{tr} U^2(t) - \mathrm{ric}(t) + \sum_{i=1}^n u_{1,i}^2(t) \\ &\leq -\operatorname{tr}\bar U^2(t) - \mathrm{ric}(t) \leq -\frac{1}{n-1}\left(\operatorname{tr}\bar U\right)^2 - \mathrm{ric}(t) = -\frac{1}{n-1}(\bar y_t')^2 - \mathrm{ric}(t).\end{aligned}$$

Setting $\bar{\mathcal{I}}_t = e^{\bar y_t + \alpha_t}$ as in **3.** we also obtain

$$\frac{d^2}{dt^2}\bar{\mathcal{I}}^{\frac{1}{N-1}} \leq -\frac{K|\dot\gamma|^2}{N-1}\bar{\mathcal{I}}^{\frac{1}{N-1}} \quad \mathrm{m}_X\text{-a.e.}$$

if $\mathrm{ric}_{M,Z}^N(\dot\gamma(t)) \geq K|\dot\gamma(t)|^2$.

5. Set $e^{\lambda_t} = L_t$. Note that (9) implies $u_{11}' \leq -u_{11}^2$. Then $L_t'' \leq 0$. By construction $\mathcal{J}_t e^{\alpha_t} = \mathcal{I}_t = \bar{\mathcal{I}}_t L_t$. Hence (similarly as in part (c) of the proof of Theorem 1.7 in [Stu06b]) we obtain

$$\mathcal{I}_t^{\frac{1}{N}} = \left(\bar{\mathcal{I}}_t L_t\right)^{\frac{1}{N}} = \left(\bar{\mathcal{I}}_t^{\frac{1}{N-1}}\right)^{\frac{N-1}{N}} (L_t)^{\frac{1}{N}} \geq \tau_{K,N}^{(1-t)}(|\dot\gamma|)\mathcal{I}_0^{\frac{1}{N}} + \tau_{K,N}^{(t)}(|\dot\gamma|)\mathcal{I}_1^{\frac{1}{N}}.$$

or

$$\mathcal{J}_t^{\frac{1}{N}} e^{\frac{1}{N}\alpha_t} \geq \tau_{K,N}^{(1-t)}(|\dot\gamma|)\mathcal{J}_0^{\frac{1}{N}} + \tau_{K,N}^{(t)}(|\dot\gamma|)e^{\frac{1}{N}\alpha_t}\mathcal{J}_1^{\frac{1}{N}} \quad \mathrm{m}_X\text{-a.e.}$$

Consider μ_0, μ_1, T_t and $\mu_t = \rho_t d\,\mathrm{vol}_g$ as in **1.**. Then for m_X-a.e. $x \in M$ we have

$$\begin{aligned}\left[\rho_t(T_t(x))e^{-\alpha_t(\gamma_x)}\right]^{-\frac{1}{N}} &= \rho_0(x)^{-\frac{1}{N}}\left[\mathcal{J}_t(x)e^{\alpha_t(\gamma_x)}\right]^{\frac{1}{N}}\\ &\geq \tau_{K,N}^{(1-t)}(|\dot\gamma_x|)\rho_0(x)^{-\frac{1}{N}}\mathcal{J}_0(x)^{\frac{1}{N}}\\ &\qquad + \tau_{K,N}^{(t)}(|\dot\gamma_x|)\left[e^{\alpha_t(\gamma_x)}\rho_0(x)^{-1}\mathcal{J}_1(x)\right]^{\frac{1}{N}}\\ &= \tau_{K,N}^{(1-t)}(|\dot\gamma_x|)\rho_0(x)^{-\frac{1}{N}}\\ &\qquad + \tau_{K,N}^{(t)}(|\dot\gamma_x|)\left[e^{-\alpha_t}\rho_1(T_1(x))\right]^{-\frac{1}{N}}.\end{aligned}$$

Recall that $\mu_t = (T_t)_\star\mu_0$. Π is the unique dynamical optimal coupling between μ_0 and μ_1. Hence, integration with respect to μ_0 yields

$$\begin{aligned}S_{N,t}^Z(\Pi) &= -\int\left[\rho_t(\gamma_t)e^{-\alpha_t(\gamma)}\right]^{-\frac{1}{N}}d\Pi(\gamma) = -\int\left[\rho_t(T_t(x))e^{-\alpha_t(\gamma_x)}\right]^{-\frac{1}{N}}d\mu_0(x)\\ &\leq -\int\left[\tau_{K,N}^{(1-t)}(|\dot\gamma_x|)\rho_0(x)^{-\frac{1}{N}} + \tau_{K,N}^{(t)}(|\dot\gamma_x|)\left[e^{-\alpha_t(\gamma_x)}\rho_1(T_1(x))\right]^{-\frac{1}{N}}\right]d\mu_0(x)\\ &= -\int\left[\tau_{K,N}^{(1-t)}(|\dot\gamma|)\rho_0(\gamma_0)^{-\frac{1}{N}} + \tau_{K,N}^{(t)}(|\dot\gamma|)e^{-\frac{1}{N}\alpha_t(\gamma)}\rho_1(\gamma_1)^{-\frac{1}{N}}\right]d\Pi(\gamma).\end{aligned}$$

That is the claim.

6. "$\Longleftarrow$": We argue by contradiction. Assume $(X, \mathrm{d}_X, \mathrm{m}_X, Z)$ satisfies the curvature-dimension condition $CD(K,N)$. We only consider the case $N > n$. Since $\sigma_{K,N}^{(t)}(\theta) \leq \tau_{K,N}^{(t)}(\theta)$, $(X, \mathrm{d}_X, \mathrm{m}_X, Z)$ satisfies the condition $CD^*(K,N)$ as well. Assume there is $v \in T_xM$ such that $\mathrm{ric}_{M,Z}^N(v) \leq (K-6\epsilon)|v|^2$. Choose a smooth function ψ such that

$$\nabla\psi(x) = v \quad \& \quad \nabla^2\psi(x) = \frac{1}{N-n}\langle Z, v\rangle|_x I_n.$$

We set $\lambda := \frac{1}{N-n}\langle Z, v\rangle|_x$. It follows $\Delta\psi(x) = -n\lambda$. We can assume that ψ has compact support in $B_1(x)$, and ψ and $-\psi$ are Kantorovich potentials by replacing ψ by $\theta\psi$ and v by θv for a sufficiently small $\theta > 0$ (we apply Theorem 13.5 in [Vil09] to ψ and $-\psi$). Then the map $T_t(y) = \exp_y(-t\nabla\psi(y))$ for $t \in [-\delta, \delta]$ induces a constant speed L^2-Wasserstein geodesic $(\mu_t)_{t\in[-\delta,\delta]}$. For this particular T_t we repeat calculations of the previous steps and obtain for $\mathcal{A}_t'\mathcal{A}_t^{-1} = U_t$

$$\mathrm{tr}U_t' + \alpha_t'' = -\mathrm{tr}U_t^2 - \mathrm{ric}(\dot\gamma_t, \dot\gamma_t) + \nabla^s Z(\dot\gamma_t, \dot\gamma_t)$$

m_X-a.e. where $\gamma_t = T_t(x)$. Now, we consider $\mu = \mathrm{m}_X(B_\eta(x))^{-1}\,\mathrm{m}_X\,|_{B_\eta(x)}$ and the induced Wasserstein geodesic $(T_t)_\star\mu = \mu_t$. We set $T_t(y) = \gamma_y(t)$, and note that $\dot\gamma_x(0) = v$. By continuity of $\mathrm{ric}_{M,Z}^N(\cdot)$ we can choose η and $\delta > 0$ such that

$$\mathrm{ric}_{M,Z}^N(\dot\gamma_y(t)) \leq (K-5\epsilon)|\dot\gamma_y(t)|^2 \text{ for } y \in B_\eta(x),\ t \in [-\delta, \delta]. \tag{11}$$

Moreover, by continuity we can choose η and $\delta > 0$ even smaller such that $\frac{1}{2}|v| \leq |\dot\gamma_y(t)|$ for $y \in B_\eta(x)$ and $t \in [-\delta, \delta]$. And again by using just continuity of Z, ψ

and all its first and second order derivatives for $\epsilon' \leq \frac{1}{4}\epsilon|v|^2$ we can choose $\eta > 0$ and $\delta > 0$ even more smaller such that

$$(a)\ |\mathrm{tr}U_t^2 - n\lambda^2| < \epsilon', \quad (b)\ \left|\frac{1}{N}(\mathrm{tr}U_t + \langle Z, \dot\gamma_t\rangle)^2 - \frac{1}{N}(n\lambda + \langle Z, \dot\gamma_t\rangle)^2\right| < \epsilon',$$
$$(c)\ \tfrac{n}{N(N-n)}\left|(N-n)\lambda - \langle Z, \dot\gamma_t\rangle\right|^2 < \epsilon'$$

for every $t \in (-\delta, \delta)$ and every $y \in B_\eta(x)$. Note that $\mathrm{tr}(\nabla^2\psi(x))^2 = n\lambda^2$ and $\Delta\psi(x) = n\lambda$. Then we compute for $y_t = \log\det\mathcal{A}_t$ - omitting the dependency on $y \in B_\eta(x)$, and using $\epsilon' \leq \epsilon\frac{1}{4}|v|^2 \leq \epsilon|\dot\gamma_y(t)|^2$ for any $y \in B_\eta(x)$ and each $t \in [-\delta, \delta]$ -

$$\begin{aligned}
y_t'' + \alpha_t'' = \mathrm{tr}U_t' + \alpha_t'' &= -\mathrm{tr}U_t^2 - \mathrm{ric}_M(\dot\gamma(t), \dot\gamma(t)) + \nabla^s Z(\dot\gamma(t), \dot\gamma(t)) \\
&\overset{(11)+(a)}{\geq} -\frac{1}{n}(n\lambda)^2 - (K-4\epsilon)|\dot\gamma_t|^2 - \frac{1}{N-n}\langle Z, \dot\gamma_t\rangle^2 \\
&= -\frac{1}{N}(n\lambda + \langle Z, \dot\gamma_t\rangle)^2 - (K-4\epsilon)|\dot\gamma_t|^2 \\
&\qquad - \frac{n}{N(N-n)}(\langle Z, \dot\gamma_t\rangle - (N-n)\lambda)^2 \\
&\overset{(c)}{\geq} -\frac{1}{N}(n\lambda + \langle Z, \dot\gamma_t\rangle)^2 - (K-3\epsilon)|\dot\gamma_t|^2 \\
&\overset{(b)}{\geq} -\frac{1}{N}(\mathrm{tr}U_t + \alpha_t')^2 - (K-2\epsilon)|\dot\gamma_t|^2 \\
&= -\frac{1}{N}(y_t' + \alpha_t')^2 - (K-2\epsilon)|\dot\gamma_t|^2 \quad \text{for } t \in [-\delta, \delta]
\end{aligned}$$

where we use

$$\frac{1}{N}(a+b)^2 + \frac{n}{N(N-n)}\left(b - a\frac{N-n}{n}\right)^2 = \frac{1}{n}a^2 + \frac{1}{N-n}b^2$$

in the third equality. Recall $\mathrm{tr}U_t = y_t'$. The previous differential inequality is equivalent to

$$\left[e^{\frac{1}{N}(y_t+\alpha_t)}\right]'' \geq -\frac{K-2\epsilon}{N}|\dot\gamma_t|^2 e^{\frac{1}{N}(y_t+\alpha_t)} \quad \text{on } [-\delta, \delta].$$

A reparametrization $s \in [-\frac{1}{2}, \frac{1}{2}] \to 2s\delta$ yields

$$\left[e^{\frac{1}{N}(\hat y_t+\hat\phi_t)}\right]'' \geq -\frac{K-2\epsilon}{N}\left|\dot{\hat\gamma}_t\right|^2 e^{\frac{1}{N}(\hat y_t+\hat\phi_t)} \quad \text{on } [-\tfrac{1}{2}, \tfrac{1}{2}]. \tag{12}$$

where $\hat y_t + \hat\phi_t = y_{2\delta t} + \phi_{2\delta t}$, $\hat\gamma_t = \exp(-2t\delta\nabla\phi)$ and $|\dot{\hat\gamma}_t| = 2\delta|\dot\gamma_t| = L(\hat\gamma)$. $\delta\phi$ is a Kantorovich potential for the Wasserstein geodesic $t \in [-\frac{1}{2}, \frac{1}{2}] \mapsto \mu_{2\delta t}$, and by the previous computation (12) is the differential inequality as in step (4) for the potential $\delta\phi$ but with reversed inequalities. We obtain (7) with reverse inequalities and K replaced by $K - 2\epsilon$ that contradicts the condition $CD^*(K, N)$. □

5. Geometric consequences

Consider a generalized smooth metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ with regular set M. In this section we assume that vol_M is m_X-absolutely continuous. We consider again a vector field Z instead of a 1-form.

Recall the definition of the measure $\mathcal{M}_{\mu,A}$ for $\mu \in \mathcal{P}^2(X)$ and $A \subset X$ measurable from section 2. For any measurable set $A \subset X$ with $\mathrm{m}_X(A) > 0$ and any point $x_0 \in$

X we define $\mathcal{M}_{x_0,A} = \Psi_\star(\delta_{x_0} \otimes \mathrm{m}_X|_A)$ where $\Psi(x,y) = \gamma_{x,y} : [0,1] \to X \in \mathcal{G}(X)$. If $x_0 \in \operatorname{supp} \mathrm{m}$ is a regular point, that is $x_0 \in M$, then $\Pi_{x_0,A} = \mathrm{m}_X(A)^{-1}\mathcal{M}_{x_0,A}$ is the unique optimal dynamical plan between δ_{x_0} and $\mathrm{m}_X(A)^{-1}\,\mathrm{m}_X|_A$. For instance, this follows, since the McCann-Brenier theorem guarantees the existence of a unique optimal map for optimal transports involving at least one vol_g-absolutely continuous measure.

We define for $r \in (0,\infty)$ and $\bar{B}_r(x_0) = \overline{B_r(x_0)}$

$$v^{Z,x_0}(r) := \int e^{\alpha_1(\gamma)}\mathcal{M}_{x_0,\bar{B}_r(x_0)}(d\gamma) = \int e^{\alpha_1(\gamma_{x,y})}d(\delta_{x_0} \otimes \mathrm{m}_X|_{\bar{B}_r(x_0)})(x,y)$$

where $\alpha_1(\gamma)$ is the line integral of Z along $\gamma : [0,1] \to X$. If $Z = 0$, then $v^{Z,x_0}(r) = \mathrm{m}_X(\bar{B}_r(x_0)) =: v^{x_0}(r)$.

We say the vector field Z has *locally finite flow* if for any $x_0 \in X$ there is $r > 0$ such that $v^{Z,x_0}(r) < \infty$. If $Z = \nabla f$ for some smooth f, Z has locally finite flow if $e^{-f}\,\mathrm{m}_X$ is a locally finite measure.

We define for $r \in (0,\infty)$

$$s^{Z,x_0}(r) := \limsup_{\delta\to 0}\delta^{-1}\int e^{\phi(\gamma_{x,y})}\delta_x \otimes \mathrm{m}_X|_{\bar{B}_{r+\delta}(x_0)\setminus B_r(x_0)}.$$

If $Z = 0$, then $s^{Z,x_0}(r) =: s^{x_0}(r)$.

Theorem 5.1 (Generalized Bishop-Gromov inequality)**.** *Let* $(X, \mathrm{d}_X, \mathrm{m}_X) = \mathrm{X}$ *be a generalized smooth metric measure space, and* Z *an* L^2*-integrable vector field. Assume* (X, Z) *satisfies the curvature-dimension* $CD(K,N)$ *for* $K \in \mathbb{R}$ *and* $N \geq 1$*. Then each bounded set* $X' \subset \operatorname{supp} \mathrm{m}_X$ *has finite* m_X*-measure, and either* m_X *is supported by one point or all points and all spheres have mass* 0*.*

Moreover, if $N > 1$ *then for each fixed* $x_0 \in \operatorname{supp}[\mathrm{m}_X] \cap M$ *and all* $0 < r < R$ *(with* $R \leq \pi\sqrt{\frac{N-1}{K}}$ *if* $K > 0$*), we have*

$$\frac{s^{Z,x_0}(r)}{s^{Z,x_0}(R)} \geq \frac{\sin^{N-1}_{K/(N-1)}(r)}{\sin^{N-1}_{K/(N-1)}(R)} \tag{13}$$

and

$$\frac{v^{Z,x_0}(r)}{v^{Z,x_0}(R)} \geq \frac{\int_0^r \sin^{N-1}_{K/(N-1)}(t)dt}{\int_0^R \sin^{N-1}_{K/(N-1)}(t)dt} \tag{14}$$

In particular, if $K = 0$*, then*

$$\frac{s^{Z,x_0}(r)}{s^{Z,x_0}(R)} \geq \frac{r^{N-1}}{R^{N-1}} \quad \& \quad \frac{v^{Z,x_0}(r)}{v^{Z,x_0}(R)} \geq \frac{r^N}{R^N}$$

for all $R, r > 0$ *and* x_0*, and the latter also holds for* $N = 1$ *and* $K \leq 0$*.*

Proof. Fix a point $x_0 \in \operatorname{supp}[\mathrm{m}_X] \cap M$, and assume first that $\mathrm{m}_X(\{x_0\}) = 0$. We set $A_R = \bar{B}_{R+\delta R}(x_0)\setminus B_R(x_0)$, and $t \in [0,1] \mapsto (e_t)_\star\Pi_{x_0,A_R} = \mu_t \in \mathcal{P}^2(X)$. $(\mu_t)_{t\in[0,1]}$ is the unique geodesic between its endpoints. Note that by definition of a generalized smooth metric measure space transport geodesics connecting $x_0 \in M$ and points in A_R are $\Pi_{x_0,A}$-almost surely contained in M. Therefore, since $x_0 \in M$ by Riemannian calculus $\mu_t \in \mathcal{P}^2(\mathrm{vol}_M)$ for any $t \in (0,1)$, and since vol_M is m_X-absolutely continuous, it follows that $\mu_t \in \mathcal{P}^2(\mathrm{m}_X)$ for any $t \in (0,1)$. In particular, $s \mapsto (e_{(1-s)t_0+st_1})_\star\Pi_{x_0,A_R} =: \mu_s$ is the unique Wasserstein geodesic

between $(e_{t_0})_\star \Pi_{x_0,R}$ and $(e_{t_1})_\star \Pi_{x_0,R}$ for any $t_0, t_1 \in (0,1]$. The unique optimal dynamical plan $\tilde{\Pi}$ is obtained by pushforward Π_{x_0,A_R} w.r.t. the map $\gamma \mapsto \tilde{\gamma}$ with $\tilde{\gamma}(s) = \gamma((1-s)t_0 + st_1)$. Set $\tilde{\rho}_s = \rho_{(1-s)t_0+st_1}$.

First, we check continuity of $t \mapsto \left[\rho_t(\gamma_t)e^{-\alpha_t(\gamma)}\right]^{-\frac{1}{N}} =: h_t(\gamma)$ on $(0,1)$. By the condition $CD^*(K,N)$ and Theorem 3.6 we have

$$\left[\tilde{\rho}_t(\gamma_t)e^{-\alpha_t(\gamma)}\right]^{-\frac{1}{N}} \geq \sigma_{K,N}^{(1-t)}(|\dot{\gamma}|)\tilde{\rho}_0(\gamma_0)^{-\frac{1}{N}} + \sigma_{K,N}^{(t)}(|\dot{\gamma}|)\left[e^{-\alpha_1(\gamma)}\tilde{\rho}_1(\gamma_1)\right]^{-\frac{1}{N}} \tag{15}$$

for $t \in [0,1]$ and $\tilde{\Pi}$-almost every $\gamma \in \mathcal{G}(X)$. That is

$$\frac{d^2}{dt^2}\left[\rho_t(\gamma_t)e^{-\alpha_t(\gamma)}\right]^{-\frac{1}{N}} \leq -\frac{K}{N}|\dot{\gamma}|^2\left[\rho_t(\gamma_t)e^{-\alpha_t(\gamma)}\right]^{-\frac{1}{N}} \quad \text{on } (0,1)$$

in distributional sense for Π-almost every γ. From (15) one can see that $h_t(\gamma)$ is C-convex for some $C < 0$ and therefore continuous on $(0,1)$ for Π-almost every γ.

Now, we choose $t_0 = \epsilon$ and $t_1 = 1$, and by the condition $CD(K,N)$ and again Theorem 3.6 we have

$$\begin{aligned}\left[\tilde{\rho}_s(\gamma_s)e^{-\alpha_s(\gamma)}\right]^{-\frac{1}{N}} &\geq \tau_{K,N}^{(1-s)}(|\dot{\gamma}|)\tilde{\rho}_0(\gamma_0)^{-\frac{1}{N}} + \tau_{K,N}^{(s)}(|\dot{\gamma}|)\left[e^{-\alpha_1(\gamma)}\tilde{\rho}_1(\gamma_1)\right]^{-\frac{1}{N}} \\ &\geq \tau_{K,N}^{(s)}(|\dot{\gamma}|)\left[e^{-\alpha_1(\gamma)}\rho_1(\gamma_1)\right]^{-\frac{1}{N}} \quad \text{for } \tilde{\Pi}\text{-a.e. } \gamma \in \mathcal{G}(X),\end{aligned}$$

or equivalently, for Π-almost every $\gamma \in \mathcal{G}(X)$:

$$\left[\rho_{(1-s)\epsilon+s}(\gamma_{(1-s)\epsilon+s})e^{-\alpha_{(1-s)\epsilon+s}(\gamma)}\right]^{-\frac{1}{N}} \geq \tau_{K,N}^{(s)}((1-\epsilon)|\dot{\gamma}|)\left[e^{-\alpha_t(\gamma)}\rho_1(\gamma_1)\right]^{-\frac{1}{N}}.$$

By continuity of $h_t(\gamma)$ in $(0,1)$, it follows for $\epsilon \to 0$

$$\rho_s(\gamma_s)^{-1}e^{\alpha_t(\gamma)} \geq \tau_{K,N}^{(s)}(|\dot{\gamma}|)^N e^{\alpha_t(\gamma)}\rho_1(\gamma_1)^{-1}.$$

Now, we choose $s = r/R$ for $r \in (0,R)$. Integration with respect to Π yields

$$\int \rho_{r/R}(\gamma_{r/R})^{-1}e^{\alpha_{r/R}(\gamma)}d\Pi(\gamma) \geq \tau_{K,N}^{(r/R)}(R \pm R\delta)^N \int e^{\alpha_t(\gamma)}d\mathcal{M}_{x_0,A_R(x_0)}(\gamma) \tag{16}$$

where we choose $-$ in τ if $K \geq 0$, and $+$ otherwise.

Note that the left hand side of the previous inequality can be rewritten as follows

$$\begin{aligned}\int \rho_{r/R}(\gamma_1)^{-1}e^{\alpha_1(\gamma)}d\bar{\Pi}(\gamma) &= \int \rho_{r/R}(\gamma_1)^{-1}e^{\alpha_1(\gamma_{x_0,\gamma_1})}d\bar{\Pi}(\gamma) \\ &= \int \rho_{r/R}^{-1}(y)e^{\alpha_t(\gamma_{x_0,y})}d(e_{r/R})_\star\Pi(y) \\ &= \int e^{\alpha_t(\Psi(x_0,y))}d\,\mathrm{m}_X\,|_{\mathrm{supp}\,\mu_{r/R}}(y) \\ &= \int e^{\alpha_t(\Psi(x,y))}d\delta_{x_0}\otimes\mathrm{m}_X\,|_{\mathrm{supp}\,\mu_{r/R}}(x,y)\end{aligned}$$

where $\bar{\Pi}$ is the optimal plan between δ_{x_0} and $(e_{r/R})_\star\Pi$. $\bar{\Pi}$ arises as the pushforward w.r.t. the map $\Phi(\gamma)(t) = \gamma(tr/R) \in \mathcal{G}(X)$. Since $\gamma_{r/R} \in A_r(x_0)$ for Π-almost every

γ, the left hand side in (16) is dominated by $\int e^{\alpha_1(\gamma)} d\mathcal{M}_{x_0,A_r(x_0)}$, and we obtain the following inequality:

$$\frac{v^{Z,x_0}(r+\delta r)-v^{Z,x_0}(r)}{\delta r} \geq \frac{\sin^{N-1}_{K/(N-1)}(r\pm r\delta)}{\sin^{N-1}_{K/(N-1)}(R\pm R\delta)} \frac{v^{Z,x_0}(R+\delta R)-v^{Z,x_0}(R)}{\delta R}. \tag{17}$$

By construction, $r \mapsto v^{Z,x_0}(r)$ is monotone non-decreasing and right continuous. Hence, it has only countably many discontinuities, and we can pick an arbitrarily small continuity point $r>0$. Then, for arbitrarily small $\delta>0$ we can apply (17) to deduce that v^{Z,x_0} is continuous at any $R>r$. Hence v^{Z,x_0} is continuous on $(0,\infty)$. In particular, for any $r>$ and $x_0\in X$ it follow that

$$\int e^{\alpha_t(\gamma)} d\mathcal{M}_{x_0,\partial B_R(x_0)}(\gamma) = \int_{\partial B_r(x_0)} e^{\alpha_t(\gamma_{x_0,y})} d\,\mathrm{m}_X(y) = 0,$$

and $\int_{\{y_0\}} e^{\alpha_t(\gamma_{x_0,y})} d\,\mathrm{m}_X(y) = 0$ for any $y_0\in X$. Hence, $\mathrm{m}_X(\partial B_r(x_0)) = \mathrm{m}_X(\{x_0\}) = 0$ for any $r>0$ and $x_0\in X$. Moreover, if $\delta\downarrow 0$ in (17), we obtain (13).

Now, for $r>0$ and $\delta>0$ given, one considers 2^n points $(1+2^{-n}\delta)r$ in $[r,(1+\delta)r)$. By (17) again it follows

$$0 \leq v^{Z,x_0}((1+2^{-n}\delta)r) - v^{Z,x_0}(r) \leq C(\delta,r)2^{-n}\delta r.$$

This implies v^{Z,x_0} is locally Lipschitz, and therefore weakly differentiable, and its weak derivative is given by $s^{Z,x_0}(r)$ $\mathcal{L}^1$-almost everywhere. Then we can apply Gromov's integration trick [Cha93, Lemma 3.1] and obtain (14).

It remains to consider the case $\mathrm{m}_X(\{x_0\})>0$. Assume $\operatorname{supp}\mathrm{m}_X \neq \{x_0\}$. Since $\operatorname{supp}\mathrm{m}_X$ is a length space, there is a least one point $y_0\in M$ such that $\mathrm{m}_X(y_0)=0$. We then apply the previous steps. This yields $\mathrm{m}_X(\{x\})=0$ for every $x\in X\setminus\{y_0\}$ which is a contradiction. Otherwise $X=\{x_0\}$ and $\mathrm{m}_X\equiv\delta_{x_0}$. □

Theorem 5.2 (Generalized Bonnet-Myers)**.** *Assume that $(X,\mathrm{d}_X,\mathrm{m}_X,Z)$ satisfies the curvature-dimension $CD(K,N)$ for $K>0$, $N\geq 1$ and Z is an L^2-integrable vector field. Then the support of m_X is compact and its diameter L satifies*

$$L \leq \pi\sqrt{\frac{N-1}{K}}.$$

In particular, if $N=1$, then $\operatorname{supp}\mathrm{m}_X$ consist of one point.

Proof. We argue by contradiction. Assume there are points $x_0,x_1\in\operatorname{supp}\mathrm{m}_X$ with $\mathrm{d}_X(x_0,x_1)>\pi\sqrt{\frac{N-1}{K}}$. Since $\mathrm{m}_X(X\backslash M)=0$, we can assume that $x_0\in M$. Choose $\epsilon>0$ such that $\theta:=\mathrm{d}_X(\delta_{x_0},\bar{B}_\epsilon(x_1))>\pi\sqrt{\frac{N-1}{K}}$, and set $\mu=\mathrm{m}_X(\bar{B}_\epsilon(x_1))^{-1}\,\mathrm{m}_X\,|_{\bar{B}_\epsilon(x_1)}$. Let $\Pi_{x_0,\mu}$ be the optimal transport between δ_{x_0} and μ. As in the proof of the Bishop-Gromov comparison $x_0\in M$ implies that $(e_t)_\star\Pi_{x_0,\mu}$ is m_X-absolutely continuous. Hence

$$\int \rho_{1/2}(\gamma_{\frac{1}{2}})^{-1} e^{\alpha_{\frac{1}{2}}(\gamma)} d\Pi(\gamma) \geq \tau^{(\frac{1}{2})}_{K,N}(\theta)^N \int e^{\alpha_t(\gamma)} d\mathcal{M}_{x_0,\bar{B}_\epsilon(x_1)}(\gamma) = \infty.$$

The left hand side is dominated by $\int e^{\alpha_1(\gamma)} d\mathcal{M}_{x_0,\bar{B}_{2\theta}(x_0)}$. Hence, $v^{Z,x_0}(2\theta)=\infty$. On the other hand, Theorem 5.1 implies that $v^{Z,x_0}(r)<\infty$ for every $r>0$. □

For $(X,\mathrm{d}_X,\mathrm{m}_X)$ and Z we define

$$\sup_{\mathcal{B}_\epsilon,\epsilon>0} \sum_{B_\epsilon(x_0)\in\mathcal{B}_\epsilon} \int e^{\alpha_1(\gamma)} d\mathcal{M}_{x_0,\bar{B}_\epsilon(x_0)}(\gamma) =: \mathrm{M}_{X,Z} \in [0,\infty]$$

where the supremum is w.r.t. families $\mathcal{B}_\epsilon$ of disjoint ϵ-balls. We also define

$$\inf_{x_0\in X}\int e^{\alpha_1(\gamma)}d\mathcal{M}_{x_0,X}(\gamma) =: \mathrm{m}_{X,Z}\in[0,\infty].$$

We have $\mathrm{M}_{X,Z}\geq \mathrm{m}_{X,Z}$, and if Z is essentially bounded, and if $\mathrm{diam}_X = D<\infty$, then $\mathrm{M}_{X,Z}\leq e^{\|Z\|_{L^\infty}D}\,\mathrm{m}_X(X)<\infty$ and $\mathrm{m}_{X,Z}\geq e^{-\|Z\|_{L^\infty}D}\,\mathrm{m}_X(X)>0$.

Let $\mathcal{X}(K,N)$ be the family of smooth metric spaces (X,d_X) such that there exits a measure m_X and an L^2-integrable vector field Z such that $(X,\mathrm{d}_X,\mathrm{m}_X,Z)$ satisfies the condition $CD(K,N)$. Let $\mathcal{X}(K,N,L)$ be the subset of metric spaces (X,d_X) in $\mathcal{X}(K,N)$ with $\mathrm{diam}_X\leq L$. Note that $\mathcal{X}(K,N)$ usually denotes the class of *metric measure spaces* that satisfy a condition $CD(K,N)$.

Corollary 5.3. *The family $\mathcal{X}(K,N,L,C)$ of smooth metric spaces (X,d_X) such that $(X,\mathrm{d}_X,\mathrm{m}_X,Z)$ satisfies $CD(K,N)$ for a measure m_X and an L^2-integrable vector field Z with*

$$\frac{\mathrm{M}_{X,Z}}{\mathrm{m}_{X,Z}}\leq C<\infty$$

is precompact with respect to Gromov-Hausdorff convergence.

Proof. Let $(X,\mathrm{d}_X)\in\mathcal{X}(K,N,L,C)$, and let m_X be a measure and Z be a vector field such that $(X,\mathrm{d}_X,\mathrm{m}_X,Z)$ satisfies $CD(K,N)$ and $\frac{\mathrm{M}_{X,Z}}{\mathrm{m}_{X,Z}}\leq C<\infty$. Fix $\epsilon>0$ and a family $\mathcal{B}_\epsilon$ of disjoint ϵ-balls in X. We replace every ϵ-ball by a slightly smaller ϵ'-ball that is contained in the first one and centered in M for some $\epsilon'<\epsilon$. This is possible since M is dense in X. The generalized Bishop-Gromov inequality yields for every $B_{\epsilon'}(x_0)$ with $x_0\in\operatorname{supp}\mathrm{m}_X\cap M$.

$$\int e^{\alpha_1(\gamma)}d\mathcal{M}_{x_0,\bar{B}_{\epsilon'}(x_0)}(\gamma)\geq C_{K,N,L}\int e^{\alpha_1(\gamma)}d\mathcal{M}_{x_0,X}(\gamma)\geq C_{K,N,L}\,\mathrm{m}_{X,Z}\,.$$

Hence $\infty>C\geq\frac{\mathrm{M}_{X,Z}}{\mathrm{m}_{X,Z}}\geq C(K,N,L)\#\mathcal{B}_\epsilon$, and the family $\mathcal{X}(K,N,L,C)$ is uniformily totally bounded, and the claim follows from Gromov's (pre)compactness theorem. □

Remark 5.4. Let us emphasize that the measures m_X do not play a role explicitly in proving that the family $\mathcal{X}(K,N,L,C)$ is uniformily totally bounded. The total masses do not need to be uniformly bounded, but the quantity $\mathrm{M}_{X,Z}/\mathrm{m}_{X,Z}$. Therefore, a sequence of measures does not have to be precompact, and consequently the statement is not formulated in terms of measured Gromov-Hausdorff convergence (or measured Gromov convergence [GMS15]). This viewpoint becomes even more apparent if one considers just vector fields Z that are uniformily essentially bounded. Then, in $\mathrm{M}_{X,Z}/\mathrm{m}_{X,Z}$ the mass cancels, and $\|Z\|_{L^\infty}\leq C$ suggests that one can extract a "weakly converging" subseqence of $\{Z\}$.

6. Examples

Riemannian manifolds with boundary. Let (M,g_M) be a geodesically convex, compact Riemannian manifold with non-empty boundary, and $\dim_M = n$. Condition (1) is obviously satisfied. Note that any Wasserstein geodesic in $\mathcal{P}^2(M)$ between absolutely continuous probability measures with bounded densities has midpoint measures with bounded densities since $\mathrm{ric}_M\geq -C$ for some $C>0$. Let Z be any smooth vector field on M that is L^2-integrable w.r.t. vol_M. Assume

that $-\nabla^s Z \geq K + \frac{1}{N} Z \otimes Z$ for some $K \in \mathbb{R}$ and $N \geq 1$. Then $(M, \mathrm{d}_M, \mathrm{vol}_M, Z)$ satisfies the condition $CD(K + K', N + n)$ if $\mathrm{ric}_M \geq K'$. In particular, if $K = 0$, $\mathrm{ric}_M \geq K' + \epsilon > 0$ and $\|Z\|_{L^\infty} \leq \sqrt{\epsilon N}$, then $\nabla^s Z \geq -\epsilon$ and $(M, \mathrm{d}_M, \mathrm{vol}_M, Z)$ satisfies the condition $CD(K', N + n)$.

N-warped products. Let (B, g_B) be a d-dimensional Riemannian manifold, and lef $f : B \to (0, \infty)$ be a smooth function. We assume

(i) $\mathrm{ric}_B \geq (d-1) K g_B$
(ii) $\nabla^2 f + K f g_B \leq 0$
(iii) $|\nabla f|^2 + K f^2 \leq K_F$.

Let $(F, g_F, \mathrm{m}_F) = \mathrm{F}$ be weighted Riemannian manifold where $\mathrm{m}_F = e^{-\Psi} d\,\mathrm{vol}_F$ is a smooth reference measure and vol_F is the Riemannian volume with respect to g_F. The Riemannian N-warped product $C := B \times_f^N F$ between B, F and f is given by the metric completion (X, d_X) of the Riemannian metric

$$g_{B \times_f F} = (p_B)^\star g_B + (f \circ p_B)^2 (p_F)^\star g_F.$$

on $B \times F$, and the measure $d\,\mathrm{vol}_B \otimes f^N d\,\mathrm{m}_F = \mathrm{m}_f^N$ on X (see also [Ket13]). $p_B : B \times F \to B$ and $p_F : B \times F \to F$ are the projection maps.

The N-Ricci tensor of $g_{B \times_f F}$ and m_f^N is computed in [Ket13]:

$$\begin{aligned}\mathrm{ric}_C^{N+d, \mathrm{m}_C^N}(\xi + v) = \mathrm{ric}_B(\xi) - N \tfrac{\nabla^2 f(\xi)}{f(p)} \\ + \mathrm{ric}_F^{N, \mathrm{m}_F}(v) - \left[\tfrac{\Delta^B f(p)}{f(p)} + (N-1) \tfrac{|\nabla f_p|^2}{f^2(p)} \right] |v|_C^2\end{aligned}$$

and the previous assumptions together with $\mathrm{ric}_F^{N, \mathrm{m}_F}(v, v) \geq (N-1) K_F |v|_F^2$ imply

$$\mathrm{ric}_C^{N+d, \mathrm{m}_C^N}(\xi + v, \xi + v) \geq (N + d - 1) K |\xi + v|_C^2$$

for every $v \in TF$ and for every $\xi + v \in TB \times_f F = TB \oplus TF$ respectively.

Let $(F, g_F) = \mathrm{F}$ be a Riemannian manifold and Z a smooth vector field on F such that (F, Z) satisfies the condition $CD(K_F(N-1), N)$. By Theorem 4.2 the condition is equivalent to $\mathrm{ric}_{F,Z}^N \geq K_F(N-1)$.

Theorem 6.1. *Let f, $(F, g_F, \mathrm{vol}_F) = \mathrm{F}$, and Z be as before, with constants $K, K_F \in \mathbb{R}$ and $N \geq 1$. Assume B has Alexandrov curvature bounded from below by K, and $f^{-1}(\{0\}) \subset \partial B$. If $N = 1$ and $K_F > 0$, we assume* $\mathrm{diam}_F \leq \pi / \sqrt{K_F}$.
Then the N-warped product $B \times_f^N F$ is a smooth metric measure space, and $(B \times_f^N F, Z^\flat)$ satisfies the condition $CD(K(N + d - 1), N + d)$ where $Z^\flat = f^{-2} Z$.

Proof. First, by a straightforward computation as in the proof of Proposition 3.2 in [Ket13] we obtain that the $N + d$-Ricci tensor of L^C is given by

$$\mathrm{ric}_{C, Z^\flat}^{N+d}(\xi + v) = \mathrm{ric}_B(\xi) - N \frac{\nabla^2 f(\xi)}{f(p)} + \mathrm{ric}_{F,Z}^N(v) - \left[\tfrac{\Delta^B f(p)}{f(p)} + (N-1) \tfrac{|\nabla f_p|^2}{f^2(p)} \right] |v|_C^2$$

and under the previous assumptions together with $\mathrm{ric}_{F,Z}^N(v) \geq (N-1) K_F |v|_F^2$, this implies

$$\mathrm{ric}_{C, Z^\flat}^{N+d,}(\xi + v) \geq (N + d - 1) K |\xi + v|_C^2.$$

Moreover, if $K_F > 0$, then by the generalized Bonnet-Myers theorem we know that

$\mathrm{diam}_F \leq \pi\sqrt{\frac{1}{K_F}}$. Note, by Theorem 1.2 in [AB04] condition (iii) from the beginning - provided the condition (ii) - is equivalent to

(1) $K_F \geq f^2 K$ if $f^{-1}(\{0\}) = \emptyset$,
(2) $K_F > 0$ and $|\nabla f| \leq \sqrt{K_F}$ if $f^{-1}(\{0\}) \neq \emptyset$.

Then, exactly like in the proof of Theorem 3.4 in [Ket13] we can prove if Π is an optimal dynamical transference plan in C such that $(e_0)_\star\Pi$ is absolutely continuous with respect to m_f^N, then

$$\Pi\left(\left\{\gamma \in \mathcal{G}(X) : \exists\, t \in (0,1) \text{ such that } \gamma(t) \in f^{-1}(\{0\})\right\}\right) = 0.$$

Hence, the N-warped product is indeed a generalized smooth metric measure space in the sense of our definition. For the proof we need a maximal diameter theorem for $(F, g_F, \mathrm{vol}_F, Z)$ that satisfies the condition $CD(K_F, N)$ with $K_F > 0$. For smooth Z this is provided by Kuwada in [Kuw13] (We also refer to [Lim10] for closely related results).

Finally, following the lines of the proof of the main theorem in [Ket13] in combination with Theorem 4.2, we obtain the condition $CD(K, N)$ for the warped product with the vector field $Z^\flat$. □

Remark 6.2. (i) In particular, the previous theorem covers the case of N-cones and N-suspensions that play an important role for the study of spaces with generalized lower Ricci curvature bounds.
(ii) One can check that $|Z^\flat|$ is L^2-integrable w.r.t. m_f^N if $N > 1$.

Example 6.3. Let $B = [0, \pi]$ and $F = \mathbb{S}^2$, and let Z be any smooth vector field on $\mathbb{S}^2$. Since Z is smooth and $\mathbb{S}^2$ is compact, we can find a constant $\kappa > 0$ such that

$$-\nabla^s Z(v,v) - \langle Z, v\rangle^2 \geq -\kappa|v|^2 \quad \text{for any } v \in T\mathbb{S}^2.$$

Now, we consider βZ with $\beta > 0$ such that $\beta\kappa \leq \frac{1}{2}$ and choose $N > 2$ such that $\frac{1}{N-2} \leq \frac{1}{\beta}$. Hence

$$-\nabla^s \beta Z(v,v) - \frac{1}{N-2}\langle \beta Z, v\rangle^2 = -\beta\left[\nabla^s Z(v,v) + \frac{\beta}{N-2}\langle Z, v\rangle^2\right] \geq -\beta\kappa|v|^2.$$

and therefore

$$\begin{aligned}\mathrm{ric}^N_{F,\beta Z}(v,v) &= \mathrm{ric}_F(v,v) - \nabla^s \beta Z(v,v) - \frac{1}{N-2}\langle \beta Z, v\rangle^2 \\ &\geq (1-\beta\kappa)|v|^2 \geq \frac{1}{2}|v|^2 = \frac{1}{2(N-2)}(N-2)|v|^2.\end{aligned}$$

Now, we set $K_F := \frac{1}{2(N-2)}$ and $f : [0,\pi] \to [0,\infty)$ with $f(r) = \sqrt{K_F}\sin(r)$. The previous theorem yields that $B \times_f^N F$ with $\frac{1}{\sin^2}\beta Z = \beta\tilde{Z}$ satisfies the condition $CD(N, N+1)$. Note $\mathrm{ric}^N_{F,Z}(v,v) = g_F(v,v)$ one only can achieve if we set $\beta = 0$. Then, any $N \geq 2$ is admissible, $Z = 0$ and, for $N = 2$, $B \times_f^2 F \simeq \mathbb{S}^3$. Otherwise, the underlying metric space has singularities at the vanishing points of f. Also note that $f^N r dr \otimes d\,\mathrm{vol}_{\mathbb{S}^2}$ is an invariant measure of the corresponding nonsymmetric diffusion operator

$$\frac{d^2}{dr^2} + \frac{N}{\sin}\frac{d}{dr} + \frac{1}{\sin^2}\left(\Delta_{\mathbb{S}^2_\beta} + \beta Z\right) = \Delta_{B\times_f^N F} + \beta\tilde{Z}$$

on $B \times_f^N F$ where $\tilde{Z} = \frac{1}{\sin^2} Z$. We observe that the diameter bound π of the generalized Bonnet-Myers theorem is attained but the N-warped product is not a smooth manifold unless $\beta = 0$ and $N = 2$! In [Kuw13] Kuwada proves that for (X, Z) (with $X = M$ a smooth manifold, $\mathrm{m}_X = \mathrm{vol}_M$, and Z a smooth vector field) that satisfies $CD(n-1, n)$ with $n \in \mathbb{N}$ and $n \geq 2$ the maximal diameter is attained if and only if $X = \mathbb{S}^n$ and $Z = 0$. We also observe that the generalized Laplace operator splits into a nonsymmetric part $\frac{\beta}{\sin^2} Z$ and a symmetric part $\Delta_{B \times_f^N F}$ where the invariant measure $f^N r dr \otimes d\,\mathrm{vol}_{\mathbb{S}^2_\beta}$ is also the invariant measure for the symmetric part, and $\int Z g f^N r dr \otimes d\,\mathrm{vol}_{\mathbb{S}^2_\beta} = 0$ for any smooth function g.

7. Evolution variational inequality and Wasserstein control

In this section we discuss the link between the curvature-dimension condition and contraction estimates in Wasserstein space. Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a compact generalized smooth metric measure space that satisfies the condition $CD(K, N)$ for a vector field Z. We assume that $X = M$, $\mathrm{m}_X = \mathrm{vol}_M$ and Z is smooth. Hence, d_X is induced by a smooth Riemannian metric g_M on M. Theorem 4.2 yields

$$\mathrm{ric}^N_{M,Z} \geq K g_M$$

for $K \in \mathbb{R}$ and $N \in [1, \infty]$. We also assume $N = \infty$.

We can consider the diffusion operator $L = \Delta + Z$ and the corresponding nonsymmetric Dirichlet form

$$\mathcal{E}(u, v) = \int \langle \nabla u, \nabla v \rangle d\,\mathrm{vol}_M + \int Z(u) v d\,\mathrm{vol}_M$$

on $L^2(\mathrm{vol}_M)$ where $D(\mathcal{E}) = W^{1,2}(X)$. Note that $\mathcal{E}$ is a Dirichlet form adapted to the Dirichlet energy w.r.t. g_M in the sense of [LSC14], and Δ is the Laplace-Beltrami operator of (M, g_M). Let P_t be the associated diffusion semigroup, let P_t^* be the so-called co-semigroup, and let $\mathcal{H}_t$ be the dual flow acting on probability measures:

$$\int (P_t u) v d\,\mathrm{vol}_M = \int u P_t^* v d\,\mathrm{vol}_M, \quad \int \phi d\mathcal{H}_t \mu = \int P_t \phi d\mu$$

where $u, v \in L^2(\mathrm{vol}_M)$ and $\phi \in C_b(X)$. L^* denotes the generator of P_t^*. Note

$$\frac{d}{dt} \int u P_t^* v d\,\mathrm{vol}_M = -\int u L^* v d\,\mathrm{vol}_M = \mathcal{E}(u, v). \tag{18}$$

For more details about nonsymmetric Dirichlet forms and the relation between P_t, P_t^*, L, L^* and $\mathcal{E}$ we refer to [MR92]. Since $\int d\mathcal{H}_t \mu = \int P_t 1 d\mu = 1$, we have $\mathcal{H}_t : \mathcal{P}^2(X) \to \mathcal{P}^2(X)$.

We assume that there is a C^∞-kernel $p_t(x, y)$ for P_t, i.e.

$$P_t u(x) = \int_X p_t(x, y) u(y) d\,\mathrm{vol}_M(y)$$

such that $p : (0, \infty) \times X^2 \to (0, \infty)$ is C^∞. Then

$$\int \phi(x) d\mathcal{H}_t \mu(x) = \int_X \int_X p_t(x, y) \phi(y) d\,\mathrm{vol}_M(y) d\mu(x)$$

for any $\phi \in C_b(X)$. In particular, $\mathcal{H}_t \mu$ is vol_M-absolutely continuous for any $\mu \in \mathcal{P}^2(M)$, and its density $\rho_t(y) = \int p_t(x, y) d\mu(x)$ is smooth. Note that we can extend P_t^* (and P_t as well) as semigroup acting on $L^1(\mathrm{vol}_M)$. Then, for vol_M-absolutely

continuous probability measures $\mu = \rho \operatorname{vol}_M$ we have $\mathcal{H}_t\mu = P_t^* \rho d \operatorname{vol}_M$. By abuse of notation we denote with L the L^1-generator of P_t, and similar for P_t^*. Then, for instance (18) extends in a canonical way provided L^∞-integrability for u. This is straightforward, and we omit details.

Proposition 7.1 (Kuwada's Lemma)**.** *Let $(M, g_M, \operatorname{vol}_M)$ and Z be as before. Let $\mu \in \mathcal{P}^2(M)$. Then $t \in (0,\infty) \mapsto \mathcal{H}_t\mu$ is a locally absolutely continuous curve in $\mathcal{P}^2(\operatorname{vol}_M)$, and*

$$|\dot{\mathcal{H}}_t\mu|^2 = \lim_{s\to 0} \frac{1}{s^2} W_2(\mathcal{H}_t\mu, \mathcal{H}_{t+s}\mu)^2 \le \int_M |\nabla \log \rho_t - Z|^2 d\mathcal{H}_t\mu. \tag{19}$$

In particular, it is differentiable almost everywhere in t.

Proof. Since $\mathcal{H}_t\mu \in \mathcal{P}^2(\operatorname{vol}_M)$ for $t > 0$ and $\mu \in \mathcal{P}^2(M)$, it is enough to consider $\mu = \rho \operatorname{vol}_M \in \mathcal{P}^2(\operatorname{vol}_M)$. Fix $t, s > 0$. By Kantorovich duality there exists a Lipschitz function φ such that

$$\frac{1}{2} W_2(\mathcal{H}_t\mu, \mathcal{H}_{t+s}\mu)^2 = \int Q_1\varphi d\mathcal{H}_{t+s}\mu - \int \varphi d\mathcal{H}_t\mu,$$

where $r \in [0,1] \to Q_r$ is the Hopf-Lax semigroup. $r \mapsto Q_r\varphi \in L^2(\operatorname{vol}_M)$ is differentiable and satisfies the Hamilton-Jacobi equation

$$\frac{d}{dr} Q_r\varphi + \frac{1}{2}|\nabla Q_r\varphi|^2 = 0$$

in the sense of [AGS14]. Since $\mathcal{H}_t\mu = \int p_t(x,\cdot) d\mu(x) = P_t^*\rho \operatorname{vol}_M$ the curve $t \in (0,\infty) \mapsto P_t^*\rho \in L^2(\operatorname{vol}_M)$ is locally Lipschitz, and therefore $r \in [0,1] \mapsto Q_r\varphi P_{t+rs}^*\rho$ is Lipschitz as well. Therefore

$$\begin{aligned}
&\int Q_1\varphi d\mathcal{H}_{t+s}\mu - \int \varphi d\mathcal{H}_t\mu = \int_0^1 \frac{d}{dr} \int Q_r\varphi P_{t+rs}^*\rho dr \\
&= -\int_0^1 \int \frac{1}{2}|\nabla Q_r\varphi|^2 d\mathcal{H}_{t+rs}\mu dr + \int_0^1 \int (Q_r\varphi) \frac{d}{dr} P_{t+rs}^*\rho d\operatorname{vol}_M\, dr.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\int (Q_r\varphi) \frac{d}{dr} P_{t+rs}^*\rho d\operatorname{vol}_M &= s \int (Q_r\varphi) L^* P_{t+rs}^*\rho d\operatorname{vol}_M \\
&= -s \int \langle \nabla Q_r\varphi, \nabla \rho_{t+rs}\rangle d\operatorname{vol}_M + s \int Z Q_r\varphi P_{t+rs}^*\rho d\operatorname{vol}_M \\
&= -s \int \langle \nabla Q_r\varphi, \frac{\nabla \rho_{t+rs}}{\rho_{t+rs}} - Z\rangle \rho_{t+rs} d\operatorname{vol}_M\,.
\end{aligned}$$

Now, recall the inequality $-\langle \nabla g, \nabla \tilde{g}\rangle \le \frac{1}{2s}|\nabla g|^2 + \frac{s}{2}|\nabla \tilde{g}|^2$. Hence

$$\int (Q_r\varphi) \frac{d}{dr} P_{t+rs}^*\rho d\operatorname{vol}_M \le \frac{1}{2} \int |\nabla Q_r\varphi|^2 d\mathcal{H}_{t+rs}\mu + \frac{s^2}{2} \int |\nabla \log \rho_{t+rs} - Z|^2 d\mathcal{H}_{t+rs}\mu$$

and consequently

$$\int Q_1\varphi d\mathcal{H}_{t+s}\mu - \int \varphi d\mathcal{H}_t\mu \le \frac{s^2}{2} \int_0^1 \int |\nabla \log \rho_{t+rs} - Z|^2 d\mathcal{H}_{t+rs}\mu dr. \tag{20}$$

By smoothness of the heat kernel p_t, $t \mapsto \mathcal{H}_{t+rs}\mu \in \mathcal{P}^2(M)$ is locally Lipschitz. Moreover, (19) easily follows from (20). □

Proposition 7.2. *Let (M, g_M, vol_M), Z and $\mathcal{H}_t\mu$ be as above. Then*

$$\frac{1}{2}\frac{d}{ds}W_2^2(\mathcal{H}_s\mu,\nu) + \frac{K}{2}W_2^2(\mathcal{H}_s\mu,\nu) \leq \int_0^1 \int \alpha(\dot\gamma)d\Pi^s(\gamma)dt + \mathrm{Ent}(\nu) - \mathrm{Ent}(\mathcal{H}_s\mu).$$

where $\nu \in \mathcal{P}^2(M)$, Π^t is the dynamical optimal plan between $\mathcal{H}_t\mu$ and ν, and α is the co-vector field that corresponds to Z.

Proof. We set $\mathcal{H}_t\mu = \mu_t = \rho_t \,\mathrm{vol}_M$, and let $(\mu_{t,r})_{r\in[0,1]}$ be the L^2-Wasserstein geodesic between $\mu_t = \mu_{t,0}$ and $\nu = \mu_{t,1} \in \mathcal{P}^2(M)$. Let $\Pi^t \in \mathcal{P}(\mathcal{G}(M))$ be the corresponding optimal dynamical plan. By the McCann-Brenier theorem [McC01] we know that Π^t and $(\mu_{t,r})_{r\in[0,1]}$ are unique and $\mu_{t,r} = \rho_{t,r}\,\mathrm{vol}_M \in \mathcal{P}^2(\mathrm{vol}_M)$ for $r \in [0,1)$. By the semigroup property of p_t it is clear that $\mathcal{H}_t\mu = P^*_{t/2}\rho_{t/2}\,\mathrm{vol}_M$. Therefore, without restriction we can assume $\mu \in \mathcal{P}^2(\mathrm{vol}_M)$. Consider

$$\mathrm{Ent}(\mu_{t,\tau}) - \int\int_0^\tau \langle Z|_{\gamma(r)}, \dot\gamma(r)\rangle dr d\Pi^t =: \mathcal{S}_\tau(\Pi^t).$$

Let $\varphi^t < \infty$ be a Kantorovich potential for the geodesic $(\mu_{t,r})_{r\in[0,1]}$. Recall that a Kantorovich potential $\varphi^t < \infty$ on X is Lipschitz. Moreover, $\rho_{t,0} \in C^\infty(X)$ since p_t is smooth. We compute exactly as in the proof of Theorem 6.5 of [AGMR15]

$$\begin{aligned}\frac{d}{d\tau}\Big|_{\tau=0}\mathcal{S}_\tau(\Pi^t) &= \frac{d}{d\tau}\Big|_{\tau=0}\mathrm{Ent}(\mu_{t,\tau}) - \frac{d}{d\tau}\Big|_{\tau=0}\int\int_0^\tau \langle Z|_{\gamma_r}, \dot\gamma_r\rangle dr d\Pi^t \\ &\geq -\int \langle \nabla\varphi^t, \nabla\rho_{t,0}\rangle d\,\mathrm{vol}_M + \int \langle Z, \nabla\varphi^t\rangle \rho_{t,0} d\,\mathrm{vol}_M\,. \end{aligned} \tag{21}$$

Note again that $\gamma_t = -\nabla\phi(\gamma_0)$ for Π-a.e. $\gamma \in \mathcal{G}(M)$. On the other hand, we can compute the derivative of the L^2-Wasserstein distance along $\mathcal{H}_t\mu$. First, by Kantorovich duality we have

$$\frac{1}{2}W_2(\mathcal{H}_t\mu,\nu)^2 = \int \varphi^t d\mathcal{H}_t\mu + \int Q_1(-\varphi^t)d\nu$$

and

$$\frac{1}{2}W_2(\mathcal{H}_{t-h}\mu,\nu)^2 \geq \int \varphi^t d\mathcal{H}_{t-h}\mu + \int Q_1(-\varphi^t)d\nu$$

Note, that we have $Q_1(-\varphi) = \varphi^c$ (we changed the sign convention for this proof). Then, for $t > 0$ and $h > 0$ we have

$$\begin{aligned}&\liminf_{h\downarrow 0}\frac{1}{h}\left(W_2(\mathcal{H}_{t-h}\mu,\nu)^2 - W_2(\mathcal{H}_t\mu,\nu)^2\right) \\ &\geq \liminf_{h\downarrow 0}\frac{1}{h}\left[\int \varphi^t d\mathcal{H}_{t-h}\mu - \int \varphi^t d\mathcal{H}_t\mu\right] \\ &= \lim_{h\downarrow 0}\frac{1}{h}\left[\int \varphi^t P^*_{t-h}\rho_0 d\,\mathrm{vol}_M - \int \varphi^t P^*_t\rho_0 d\,\mathrm{vol}_M\right] \\ &= -\int \varphi^t \frac{d}{dt}P^*_t\rho_0 d\,\mathrm{vol}_M = -\int \varphi^t L^* P^*_t \rho_0 d\,\mathrm{vol}_M \\ &= \int \langle \nabla\varphi^t, \nabla\rho_{t,0}\rangle d\,\mathrm{vol}_M - \int \langle Z|_x, \nabla\varphi^t(x)\rangle \rho_{t,0}(x) d\,\mathrm{vol}_M\,.\end{aligned}$$

The last equality follows from the relation between $\mathcal{E}$ and its dual generator L^* [MR92]. Therefore

$$\frac{d^-}{ds}\Big|_{s=t} W_2(\mathcal{H}_t\mu,\nu)^2 \le -\int \langle \nabla\varphi^t, \nabla\rho_{t,0}\rangle d\,\mathrm{vol}_M + \int \langle Z|_x, \nabla\varphi^t(x)\rangle \rho_{t,0}(x) d\,\mathrm{vol}_M,$$

where $\frac{d^-}{ds}\Big|_{s=t} f(t) = \limsup_{h\uparrow 0} \frac{1}{h}[f(t+h)-f(t)]$. Together with (21) and since $t \mapsto \mathcal{H}_t\mu$ is absolutely continuous, it follows

$$\frac{d}{d\tau}\Big|_{\tau=0} \mathcal{S}_\tau(\Pi) \ge \frac{d}{ds}\Big|_{s=t} W_2(\mathcal{H}_s\mu,\nu)^2. \tag{22}$$

The curvature-dimension condition implies that for $\mathcal{H}_t\mu$, ν and Π with $(e_r)_\star\Pi = \mu_r$

$$\begin{aligned}\mathrm{Ent}(\mu_t) - \alpha_t(\Pi) \le (1-t)\,\mathrm{Ent}(\mu_0) + t\,[\mathrm{Ent}(\mu_1) - \alpha_1(\Pi)] \\ - \frac{1}{2}Kt(1-t)KW_2(\mu_0,\mu_1)^2,\end{aligned}$$

where $\alpha_t(\Pi) = \int \alpha_t(\gamma) d\Pi(\gamma)$. Now we substract $\mathrm{Ent}(\mu_0)$, devide by $t > 0$, and let $t \to 0$. We obtain

$$\frac{d}{d\tau}\Big|_0 S_\tau(\Pi) \le -\mathrm{Ent}(\mathcal{H}_t\mu) + [\mathrm{Ent}(\nu) - \alpha_1(\Pi)] - \frac{1}{2}KW_2(\mathcal{H}_t\mu,\nu)^2.$$

Together with (22) this implies the result. □

Definition 7.3. Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space, and let Z be an L^2-integrable vector field. Let $K \in \mathbb{R}$ and $(\mu_t)_{t\in(0,\infty)} \subset \mathcal{P}^2(\mathrm{vol}_M)$ be locally absolutely continuous. We say μ_t is an $EVI_{K,\infty}$-flow curve of Z starting in $\mu_0 \in \mathcal{P}^2(X)$ if $\lim_{t\to 0}\mu_t = \mu_0$, and for every $\nu \in \mathcal{P}^2(\mathrm{vol}_M)$ the *evolution variational inequality*

$$\frac{1}{2}\frac{d}{ds}W_2^2(\mathcal{H}_s\mu,\nu) + \frac{K}{2}W_2^2(\mathcal{H}_s\mu,\nu) \le -\int_0^1 \int Z(\dot\gamma) d\Pi^s(\gamma) dr + \mathrm{Ent}(\nu) - \mathrm{Ent}(\mathcal{H}_s(\mu)).$$

holds for a.e. $t > 0$ where Π^s is the dynamical optimal plan between $\mathcal{H}_s\mu$ and ν.

Corollary 7.4. *Assume $(M, g_M, \mathrm{vol}_M, Z)$ as above satisfies the condition $CD(K,N)$. Then $t \mapsto \mathcal{H}_t\mu$ is an $EVI_{K,\infty}$ gradient flow curve for every $\mu \in \mathcal{P}^2(M)$.*

The following contraction estimate for nonsymmetric diffusions was known before.

Corollary 7.5. *Let (M, g_M, vol_M) and Z be as before. Then*

$$W_2(\mathcal{H}_t\mu, \mathcal{H}_t\nu)^2 \le e^{-2Kt} W_2(\mu,\nu)^2.$$

for $\mu, \nu \in \mathcal{P}^2(M)$.

Proof. First, note that by the previous corollary and Lemma 4.3.4 in [AGS08] the function $t \in (0,\infty) \mapsto W_2(\mathcal{H}_t\mu, \mathcal{H}_t\nu)^2$ is locally absolutely continuous, and therefore differentiable almost everywhere.

Let $t < s$. Integrating the EVI_K inequality from t to s with $\nu = \mathcal{H}_s\mu$ yields

$$\begin{aligned}&\frac{1}{2}W_2^2(\mathcal{H}_s\mu,\mathcal{H}_s\nu) - \frac{1}{2}W_2^2(\mathcal{H}_t\mu,\mathcal{H}_s\nu) + \int_t^s \frac{K}{2}W_2^2(\mathcal{H}_\tau\mu,\mathcal{H}_s\nu)d\tau \\ &\le -\int_t^s\int_0^1\int Z(\dot\gamma)d\Pi^\tau(\gamma)drd\tau + (s-t)\,\mathrm{Ent}(\mathcal{H}_s\nu) - \int_t^s \mathrm{Ent}(\mathcal{H}_\tau(\mu))d\tau.\end{aligned}$$

where Π^τ is the geodesic between $\mathcal{H}_\tau\mu$ for $\tau \in [t,s]$ and $\mathcal{H}_s\nu$, and similar if we set $\nu = \mathcal{H}_t\mu$, we obtain

$$\begin{aligned}&\frac{1}{2}W_2^2(\mathcal{H}_s\mu,\mathcal{H}_t\nu) - \frac{1}{2}W_2^2(\mathcal{H}_t\mu,\mathcal{H}_t\nu) + \int_t^s \frac{K}{2}W_2^2(\mathcal{H}_t\mu,\mathcal{H}_\tau\nu)d\tau \\ &\leq -\int_t^s\int_0^1\int Z(\dot\gamma)d\Pi^{\tau,-}(\gamma)drd\tau + (s-t)\operatorname{Ent}(\mathcal{H}_t\mu) - \int_t^s \operatorname{Ent}(\mathcal{H}_\tau(\nu))d\tau.\end{aligned}$$

where $\Pi^{\tau,-}$ is the geodesic between $\mathcal{H}_\tau\nu$ for $\tau \in [t,s]$ and $\mathcal{H}_t\mu$. Adding the previous inequalities from each other yields

$$\begin{aligned}&\frac{1}{2}W_2^2(\mathcal{H}_s\mu,\mathcal{H}_s\nu) - \frac{1}{2}W_2^2(\mathcal{H}_t\mu,\mathcal{H}_t\nu) + \frac{K}{2}\int_t^s \left[W_2^2(\mathcal{H}_t\mu,\mathcal{H}_\tau\nu) + W_2^2(\mathcal{H}_\tau\mu,\mathcal{H}_s\nu)\right]d\tau \\ &\leq -\int_t^s\int_0^1\int Z(\dot\gamma)d\Pi^{\tau}(\gamma)dtd\tau - \int_t^s\int_0^1\int Z(\dot\gamma)d\Pi^{\tau,-}(\gamma)dtd\tau \\ &\qquad + (s-t)(\operatorname{Ent}(\mathcal{H}_s\mu) + \operatorname{Ent}(\mathcal{H}_t\mu)) - 2\int_t^s \operatorname{Ent}(\mathcal{H}_\tau(\nu))d\tau.\end{aligned}$$

Dividing by $(s-t)$, and letting $s \to t$ yields

$$\begin{aligned}\frac{d}{d\tau}\Big|_{\tau=t}\frac{1}{2}W_2^2(\mathcal{H}_\tau\mu,\mathcal{H}_\tau\nu) &\leq -KW_2^2(\mathcal{H}_t\mu,\mathcal{H}_t\nu) \\ &\quad - \int_0^1\left(\int Z(\dot\gamma)d\Pi^t(\gamma) + \int Z(\dot\gamma)d\Pi^{t,-}(\gamma)\right)dt. \qquad (23)\end{aligned}$$

Since geodesic between absolutely continuous probability measures are unique, and since $\Pi^{\tau,-}$ coincides with Π^τ up to reverse parametrization, we have for $\Psi(\gamma) = \gamma^-$

$$\int Z(\dot\gamma)d\Pi^t(\gamma) = \int Z(\dot\gamma^-)d\Psi_\star\Pi^t(\gamma) = -\int Z(\dot\gamma)d\Pi^{t,-}(\gamma)$$

and the last line in (23) vanishes. Hence

$$\frac{d}{dt}\Big|_{t=0}W_2(\mathcal{H}_t\mu,\mathcal{H}_t\nu)^2 \leq -2KW_2(\mathcal{H}_t\mu,\mathcal{H}_t\nu)^2.$$

Finally, Gromwall's lemma yields the claim. □

Corollary 7.6 (Kuwada, [Kuw10])**.** *For $f \in D(\mathcal{E})$, we have*

$$|\nabla P_t f|^2 \leq e^{-2Kt}P_t|\nabla f|^2.$$

That is precisely that P_t satisfies the Bakry-Emery curvature-dimension condition $BE(K,\infty)$.

Remark 7.7. By a classical result of Bakry and Emery (for instance [Bak94]) the condition $BE(K,\infty)$ again implies $\mathrm{ric}^Z_{M,\infty} \geq K$ and therefore the condition $CD(K,\infty)$.

As consequence of Remark 7.7 we obtain Theorem 1.1.

Remark 7.8. It is rather obvious how to improve the previous estimates in context of the condition $CD(K,N)$ for $N < \infty$ in the sense of [EKS15] by modifying the computations (see also [Kuw15]), and we omit details.

References

[AB04] Stephanie B. Alexander and Richard L. Bishop, *Curvature bounds for warped products of metric spaces*, Geom. Funct. Anal. **14** (2004), no. 6, 1143–1181. MR 2135163 (2007d:53061)

[AGMR15] Luigi Ambrosio, Nicola Gigli, Andrea Mondino, and Tapio Rajala, *Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure*, Trans. Amer. Math. Soc. **367** (2015), no. 7, 4661–4701. MR 3335397

[AGS08] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré, *Gradient flows in metric spaces and in the space of probability measures*, second ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008. MR 2401600 (2009h:49002)

[AGS14] ———, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math. **195** (2014), no. 2, 289–391. MR 3152751

[Bak94] Dominique Bakry, *L'hypercontractivité et son utilisation en théorie des semigroupes*, Lectures on probability theory (Saint-Flour, 1992), Lecture Notes in Math., vol. 1581, Springer, Berlin, 1994, pp. 1–114. MR 1307413 (95m:47075)

[BS10] Kathrin Bacher and Karl-Theodor Sturm, *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces*, J. Funct. Anal. **259** (2010), no. 1, 28–56. MR 2610378 (2011i:53050)

[BS14] ———, *Ricci bounds for euclidean and spherical cones*, Singular Phenomena and Scaling in Mathematical Models (2014), 3–23.

[CEMS01] Dario Cordero-Erausquin, Robert J. McCann, and Michael Schmuckenschläger, *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb*, Invent. Math. **146** (2001), no. 2, 219–257. MR 1865396 (2002k:58038)

[CEMS06] ———, *Prékopa-Leindler type inequalities on Riemannian manifolds, Jacobi fields, and optimal transport*, Ann. Fac. Sci. Toulouse Math. (6) **15** (2006), no. 4, 613–635. MR 2295207

[Cha93] Isaac Chavel, *Riemannian geometry—a modern introduction*, Cambridge Tracts in Mathematics, vol. 108, Cambridge University Press, Cambridge, 1993. MR 1271141

[EKS15] Matthias Erbar, Kazumasa Kuwada, and Karl-Theodor Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces*, Invent. Math. **201** (2015), no. 3, 993–1071. MR 3385639

[GH] Nicola Gigli and Bang-Xian Han, *The continuity equation on metric measure spaces*, http://arxiv.org/abs/1406.6350.

[Giga] Nicola Gigli, *Nonsmooth differential geometry - An approach tailored for spaces with Ricci curvature bounded from below*, http://arxiv.org/abs/1407.0809.

[Gigb] ———, *On the differential structure of metric measure spaces and applications*, http://arxiv.org/abs/1205.6622.

[GMS15] Nicola Gigli, Andrea Mondino, and Giuseppe Savaré, *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 5, 1071–1129. MR 3477230

[GRS] Nicola Gigli, Tapio Rajala, and Karl-Theodor Sturm, *Optimal maps and exponentiation on finite dimensional spaces with Ricci curvature bounded from below*, http://arxiv.org/abs/1305.4849.

[Ket13] Christian Ketterer, *Ricci curvature bounds for warped products*, J. Funct. Anal. **265** (2013), no. 2, 266–299. MR 3056704

[Kuw10] Kazumasa Kuwada, *Duality on gradient estimates and Wasserstein controls*, J. Funct. Anal. **258** (2010), no. 11, 3758–3774. MR 2606871

[Kuw13] ———, *A probabilistic approach to the maximal diameter theorem*, Math. Nachr. **286** (2013), no. 4, 374–378. MR 3028781

[Kuw15] ———, *Space-time Wasserstein controls and Bakry–Ledoux type gradient estimates*, Calc. Var. Partial Differential Equations **54** (2015), no. 1, 127–161. MR 3385156

[Lim10] Murat Limoncu, *Modifications of the Ricci tensor and applications*, Arch. Math. (Basel) **95** (2010), no. 2, 191–199. MR 2674255

[LSC14] Janna Lierl and Laurent Saloff-Coste, *The Dirichlet heat kernel in inner uniform domains: local results, compact domains and non-symmetric forms*, J. Funct. Anal. **266** (2014), no. 7, 4189–4235. MR 3170207

[LV09] John Lott and Cédric Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2) **169** (2009), no. 3, 903–991. MR 2480619 (2010i:53068)

[McC01] Robert J. McCann, *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal. **11** (2001), no. 3, 589–608. MR 1844080 (2002g:58017)

[MR92] Zhi Ming Ma and Michael Röckner, *Introduction to the theory of (nonsymmetric) Dirichlet forms*, Universitext, Springer-Verlag, Berlin, 1992. MR 1214375

[OV00] F. Otto and C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal. **173** (2000), no. 2, 361–400. MR 1760620 (2001k:58076)

[RS14] Tapio Rajala and Karl-Theodor Sturm, *Non-branching geodesics and optimal maps in strong $CD(K,\infty)$-spaces*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 831–846. MR 3216835

[Stu06a] Karl-Theodor Sturm, *On the geometry of metric measure spaces. I*, Acta Math. **196** (2006), no. 1, 65–131. MR 2237206 (2007k:53051a)

[Stu06b] ———, *On the geometry of metric measure spaces. II*, Acta Math. **196** (2006), no. 1, 133–177. MR 2237207 (2007k:53051b)

[Vil09] Cédric Villani, *Optimal transport, old and new*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009. MR 2459454 (2010f:49001)

[vRS05] Max-K. von Renesse and Karl-Theodor Sturm, *Transport inequalities, gradient estimates, entropy, and Ricci curvature*, Comm. Pure Appl. Math. **58** (2005), no. 7, 923–940. MR 2142879 (2006j:53048)

[Wan11] Feng-Yu Wang, *Equivalent semigroup properties for the curvature-dimension condition*, Bull. Sci. Math. **135** (2011), no. 6-7, 803–815. MR 2838102 (2012j:58052)