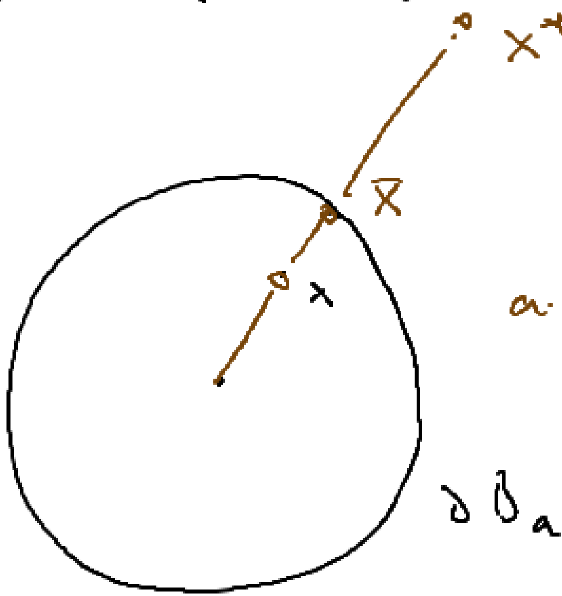
Last time:

Let Ω be open with smooth bdy.

Recall Def of Green fct.

<u>Green fct of a ball $B_a(0)$</u> $= \{x \in \mathbb{R}^3 : |x| < a\} = B_a$

$$x^* = \frac{a}{|x|} \cdot \bar{x} = \frac{a^2}{|x|^2} \cdot x$$



$$|x| \cdot |x^*| = a^2$$

$$a \cdot \frac{x}{|x|} = \bar{x}$$

Theorem: $H^y(x) = \frac{a}{|y|} \cdot \frac{1}{4\pi} \cdot \frac{1}{|x-y^*|}$

solves $\Delta H^y = 0$ on B_a

$$H^y|_{\partial B_a} = -\Phi_3(x-y) = \frac{1}{4\pi}\frac{1}{|x-y|} \quad x \in \partial B_a$$

Hence the green fct of B_a is

$$G(x,y) = \Phi_3(x-y) + H^y(x) = \frac{1}{4\pi}\left(\frac{-1}{|x-y|} + \frac{a}{|y|}\frac{1}{|x-y^*|}\right)$$

Proof: ① $H^y \in C^\infty(\overline{B_a})$

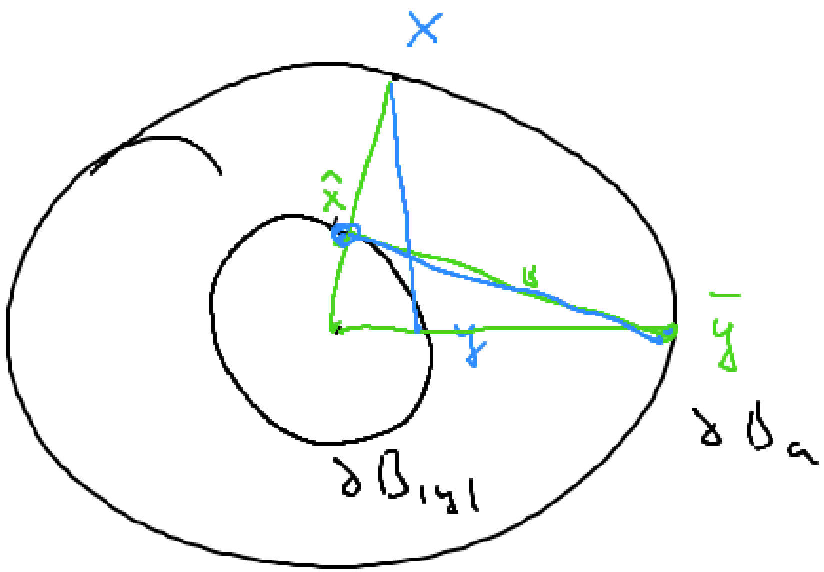
② $\Delta H^y = 0$ on $\overline{B_a}$

③ $x \in \partial B_a \quad |x| = a$

Claim: $\frac{|y|}{a} \cdot |x-y^*| = |x-y| \implies H^y|_{\partial B_a} = -\phi(x-y)$

Proof of claim. $x \in \partial B_a \quad y \in B_a \quad y \neq 0$

$$\frac{|y|}{a}|x - y^*| = \Big|\underbrace{\frac{|y|}{a}x}_{\hat{x}} - \underbrace{\frac{|y|}{a}\cdot y^*}_{\bar{y}}\Big| = |x - y|$$



$$y^* = \frac{a}{|y|}\bar{y} = \frac{a^2}{|y|^2}y$$

$\boxed{}$

Corollary Let $u \in C^2(\overline{B_a})$ a solution of $\Delta u = 0$ on $\overline{B_a}$

$$u|_{\partial B_a} = h$$

then $$u(y) = \frac{a^2 - |y|^2}{4\pi a} \int_{\partial B_a} \frac{h(y)}{|x-y|^3} \, ds(y). \quad y \in B_a$$

Proof: $$u(y) = \int_{\partial B_a} h(x) \cdot \frac{\partial G(\cdot, y)}{\partial N}\Big|_x \, ds(x)$$

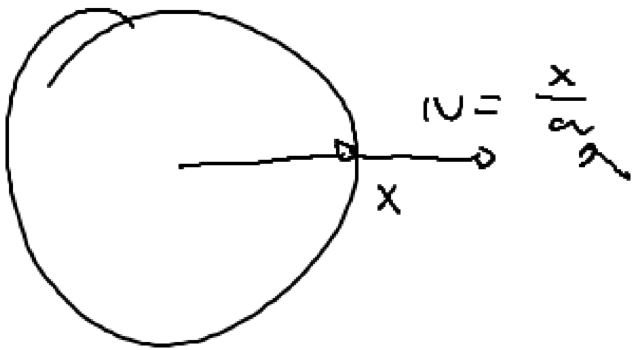
$$\nabla G(\cdot, y)|_x = \nabla\left(\frac{-1}{4\pi}\frac{1}{|x-y|} + \frac{1}{4\pi}\frac{a}{|y|}\frac{1}{|x-y^*|}\right)$$

$$= + \frac{1}{4\pi}\frac{1}{|x-y|^3}(x-y) - \frac{1}{4\pi}\frac{1}{|x-y^*|^3}\frac{a}{|y|}(x-y^*)$$

$$= \frac{1}{4\pi}\frac{1}{|x-y|^3}\left(x - y - \frac{|y|^2}{a^2}x + \underbrace{\frac{|y|^2}{a^2}\cdot y^*}_{y}\right)$$

$$= \frac{1}{4\pi}\frac{1}{|x-y|^3}\left(x - \frac{|y|^2}{a^2}x\right)$$

$N = \frac{x}{a}$ $\qquad \frac{\partial G(\cdot, y)}{\partial N} = \langle \nabla G(\cdot, y), N \rangle$



$$= \frac{1}{4\pi a}\frac{1}{|x-y|^3}\Big(\underbrace{\langle x, x\rangle}_{a^2} - \underbrace{\langle x, \tfrac{|y|^2}{a^2} x\rangle}_{|y|^2}\Big)$$

$$= \frac{1}{4\pi a}\,\frac{a^2 - |y|^2}{|x-y|^3} \qquad \blacksquare$$

<u>Remark</u> $n \geq 2$

$$G(x,y) = \frac{-1}{n(n-2)\,\mathrm{vol}^n(B_1)}\left(\frac{1}{|x-y|^{n-2}} - \frac{a}{|y|}\,\frac{1}{|x-y^*|^{n-2}}\right)$$

is Green fct of $B_a \subset \mathbb{R}^n$

and similar if u solves $\Delta u = 0$ $\overline{B_a}$

$$u(y) = \frac{1}{n\,\mathrm{vol}(B_1)}\,\frac{a^2 - |y|^2}{1}\int_{\partial B_a}\frac{u(x)}{|x-y|^n}\,ds(x).$$

<u>Theorem</u> Let $h \in C^0(\partial B_a)$. Define

$$u(y) = \begin{cases} \dfrac{a^2 - |y|^2}{n \cdot \mathrm{vol}(B)} \cdot \displaystyle\int_{\partial B_a} \dfrac{h(x)}{|x-y|^n}\, ds(x) & y \in B_a \\ h(y) & y \in \partial B_a \end{cases}$$

Then $u \in C^2(B_a) \cap C^0(\overline{B_a})$ and solves

$$\Delta u = 0 \quad \text{on } B_a$$

$$u|_{\partial B_a} = h$$