MAT351 Partial Differential Equations **Fall, Assignment 0**

Christian Ketterer September 14, 2020

Assignments

Read Sections 2.1-2.9 in *Partial Differential Equations: A First Course* by R. Choksi.

Problems

1. *Classification of PDEs*

   (a) For each of the following PDEs answer the next questions. What is the order of the PDE? Is the PDE linear, semi-linear, quasi-linear or fully non-linear? If the PDE is linear, is it homogeneous or non-homogeneous?

      i. $u_t + u_x = 0$ *(Simple transport equation)*,
      ii. $u_y y + y u_x x - u^2 = 0$,
      iii. $u_t + (u_x)^2 = 0$, *(Hamilton-Jacobi equation)*,
      iv. $u_{xx} + u_{yy} + u_{zz} = f$ *(Poisson equation)*,
      v. $u_x + uu_x + u_{xxx} = 0$ *(Korteveg-de-Vries equation)*,
      vi. $u_t + \Delta(u^m) = 0$ *(Porous medium equation)*,
      vii. $u_{tt} + u_{xxxx} = 0$ *(Beam equation)*,
      viii. $u_t + \sigma x^2 u_{xx} + r x u_x + r u = 0$ for constants σ, r *(Black Scholes equation)*,
      ix. $\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0$ *Minimal surface equation.*

   (b) Read online in which physical, economical or mathematical context the previous PDEs are relevant. In particular, what do the dependent and independent variable describe, and what do the parameters in the PDE mean?

2. *Fudamental solution for the Laplace equation*

   (a) Show that for every $\epsilon > 0$

   $$u(x, y) = \log \sqrt{x^2 + y^2}$$

   solves

   $$\Delta u = 0 \text{ on } \Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > \epsilon\}$$

   (b) Show that for every $\epsilon > 0$

   $$u(\mathbf{x}) = \frac{1}{|\mathbf{x}|_2^2}$$

solves

$$\Delta u = 0 \quad \text{on} \quad \Omega = \{\mathbf{x} \in \mathbb{R}^4 : |\mathbf{x}|_2^2 > \epsilon\}$$

where $|\mathbf{x}|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$.

3. *Divergence theorem*

   Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vectorfield with $|V(\mathbf{x})|_2 \leq \frac{1}{1+|\mathbf{x}|_2^3}$. Show that

$$0 = \int_{\mathbb{R}^3} \nabla \cdot V d\mathbf{x}.$$

   (Hint: Apply the divergence theorem on large domains.)