MAT351 Partial Differential Equations **Fall, Assignment 1**

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Hand-in Problems (Due till September 28 before lecture, via crowdmark)

H1 *Fundamental theorem of calculus of variations.* Let $f \in C^0(\mathbb{R}^n)$ such that

$$\int_\Omega f(\mathbf{x})d\mathbf{x} = 0$$

for any domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Show that $f \equiv 0$.

Conclude that, if $f, g \in C^0(\mathbb{R}^n)$, then

$$\int_\Omega f(\mathbf{x})d\mathbf{x} = \int_\Omega g(\mathbf{x})d\mathbf{x}$$

for any domain $\Omega \subset \mathbb{R}^n$ with smooth boundary implies $f \equiv g$.

H2 Find the general solution of

$$u_x + (1 + y^2)u_y = 0.$$

Sketch some of the characteristics.

Problems for discussion

1. *Wellposed problem in linear algebra.* Given an $n \times m$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^n$ we consider the linear system

$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

   (a) Write down precise statements for existence, uniquness and stability.

   (b) Which properties of A imply existence and uniqueness of a solution $\mathbf{x} \in \mathbb{R}^m$ for (1)?

   (c) Which property of A ensures stability?

   (d) What does stability has to do with the modulus of the smallest eigenvalue?

2. *Boundary conditions.* Recall that Fourier's law says that heat flows from hot to cold regions, proportional to the temperature gradient.

   (a) Show that the temperature of a rod in $[0, \infty) \times \mathbb{R}^2$ that is insulated at one end at $x = 0$ ($(x, y, z) \in \mathbb{R}^3$), satisfies the boundary condition $\frac{\partial u}{\partial x} = 0$. *(Hint: Use Fourier's law.)*

   (b) Do the same for the diffusion of a gas along a tube that is closed off at the end $x = 0$. *(Hint: Use Fick's law.)*

3. *Steady-state temperature.* A homogeneous body that occupies a solid bounded domain Ω is completely insulated. Its temperature at time 0 is given by $f(\mathbf{x}), \mathbf{x} \in \Omega$. Find the steady-state temperature distribution that is reached after a long time. *(Hint: No heat is lost.)*

To Read

1. Sections 1.3-1.5 in *Partial Differential Equations* by W. Strauss
2. Section 4.1 and 4.2 in *Partial Differential Equations: A First Course* by R. Choksi.