MAT351 Partial Differential Equations **Fall, Assignment 10**

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Hand-in Problems (Due till January 18 before lecture, via crowdmark)

1. The Dirichlet problem for the exterior of a circle is

$$\begin{aligned} u_{x,x} + u_{y,y} &= 0 \quad \text{on} \quad \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > a^2\} \\ u &= h \quad \text{on} \quad \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = a^2\} \\ \limsup_{x^2+y^2\to\infty} |u(x,y)| &\le C < \infty \end{aligned}$$

   Derive the Poisson formula in polar coordinates for this problem:

$$\tilde{u}(r,\theta) = \frac{r^2 - a^2}{2\pi} \int_0^{2\pi} \frac{\tilde{h}(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} d\phi$$

   where $\tilde{h}(\theta) = h(a\cos\theta, a\sin\theta)$.
   Hint: Follow the derivation of the Poisson formula on the disk $B_a(0)$.

2. Let $B_1 \subset \mathbb{R}^2$ be the unit disc and let $B_1^+ = B_1 \cap \{(x,y) \in \mathbb{R}^2 : y > 0\}$. Let u be a function that is harmonic on B_1^+ and continuous on $\overline{B_1^+}$. Assume that u vanishes on $\overline{B_1^+} \cap \{(x,y) \in \mathbb{R}^2 : y = 0\} = \{(x,0) \in \mathbb{R}^2 : x \in [-1,1]\}$. Consider the extension of u to the whole disc $\overline{B_1}$ by odd reflection

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & \text{if } (x,y) \in \overline{B_1} \text{ and } y \ge 0, \\ -u(x,-y) & \text{if } (x,y) \in \overline{B_1} \text{ and } y < 0. \end{cases}$$

   Prove that $\tilde{u}$ is harmonic by identifying $\tilde{u}$ as the solution of a suitable boundary-value problem.
   Hint: You will need to use uniqueness twice.

Problems for discussion

1. Suppose that u is harmonic on the disk $B_2 \subset \mathbb{R}^2$ and that $u(2\cos\theta, 2\sin\theta) = 2021 + (\sin\theta)^{17}$. Without computing the solution, find
   (a) the maximum on $\overline{B_2}$;
   (b) the value of u in the origin;
   (c) the integral of u over the disk.

2. Let $f \in C^0(\mathbb{R})$ be 2π-periodic. Show that the full Fourier series of f converges to f in the mean square sense if and only if *Parseval's equality* holds:

$$\frac{\pi}{2} A_0^2 + \sum_{n=1}^{\infty} \left(|A_n|^2 \int_0^{2\pi} \cos(nx) dx + |B_n|^2 \int_0^{2\pi} \sin(nx) dx \right) = \int_0^{2\pi} f(x) dx.$$

   Hint: Least square approximation.

3. Let f be a smooth 2π-periodic function with $\int_{-\pi}^{\pi} f(x)dx = 0$. Use Parseval's identity to show that

$$\sqrt{\int_0^{2\pi} (f(x))^2 dx} = \|f\|_2 \leq \|f'\|_2 .$$

4. (a) Recall the full Fourier series of $f(x) = x$ on $(-\pi, \pi)$. Apply Parseval's inequality to find the value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (b) Recall the full Fourier series of $f(x) = x^2$ on $(-\pi, \pi)$. Apply Parseval's inequality to find the value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

To Read

1. Section 7.1, 7.2, 7.3 in *Partial Differential Equations: An Introduction* by W. Strauss.