MAT351 Partial Differential Equations **Fall, Assignment 11**

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Hand-in Problems (Due till January 25 before lecture, via crowdmark)

1. (a) Show that

$$\Phi_n(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } n = 2, \\ -\frac{1}{n(n-2)\text{vol}(B_1(0))}\frac{1}{|x|^{n-2}} & \text{if } n \in \mathbb{N}, n \geq 3. \end{cases}$$

solves the Laplace equation in $\mathbb{R}^n \backslash \{0\}$.

(b) Given $f \in C^2(\mathbb{R}^3)$ with compact support show that

$$u(x) = \int_{\mathbb{R}^n} \Phi_n(y - x)f(y)dy$$

solves the Poisson equation $\Delta u = f$ on $\mathbb{R}^3$.

Hint: Follow the proof of the Representation formula.

2. Let $\Omega \subset \mathbb{R}^n$ be open with smooth boundary. Show that

   (a) Green's function $G(x, y)$, $x, y \in \Omega$, is uniquely determined by its properies,

   (b) $G(x, y) < 0$, $x, y \in \Omega$.

Problems for discussion

1. Derive the mean value property for a harmonic function u on $\Omega \subset \mathbb{R}^3$ (with smooth boundary) from the representation formula

$$u(x) = \int_{\partial\Omega} \Big[- u(y) \frac{\partial}{\partial N}\Big|_y \left(\frac{1}{|\cdot - x|}\right) + \frac{1}{|y - x|}\frac{\partial u}{\partial N}\Big|_y \Big] ds(y).$$

2. Let $\Omega \subset \mathbb{R}^n$ be open with smooth boundary. A function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves the Neumann Problem for the Laplace equation if

$$\Delta u = 0 \text{ on } \Omega,$$
$$\frac{\partial u}{\partial N} = 0 \text{ on } \partial\Omega$$

where N is the unit normal vector field on $\partial\Omega$. The kinetic energy of u is

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx.$$

Use the energy method to show uniqueness up to addition of constants of the Neumann problem.

3. Find the 1-dimensional Green's function $G(x, x_0)$ for the interval $(0, l)$ defined by the following 3 properties:

   (i) $\frac{d}{dx} G(x, x_0) = 0$, $x \neq x_0$,

   (ii) $G(0, x_0) = G(l, x_0) = 0$,

   (iii) $G(x)$ is continuous a x_0 and $G(x, x_0) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.

To Read

1. Section 7.4 in *Partial Differential Equations: An Introduction* by W. Strauss.