MAT351 Partial Differential Equations **Fall, Assignment 12**

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Hand-in Problems (Due till February 22 before lecture, via crowdmark)

1. (*Lorentz invariance of the wave equation*) We denote points in $\mathbb{R}^4$ with $(x, y, z, t) \in \mathbb{R}^3 \times \mathbb{R}$. Let

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

A matrix L is called a *Lorentz transformation* if L has an inverse L^{-1} and $L^{-1} = \Gamma L^T \Gamma$, where L^T is the transpose of L.

(a) If L and M are Lorentz, show that LM and L^{-1} also are.

(b) Show that L is Lorentz if and only if $m(Lv) = m(v)$ for all 4-vectors $v = (x, y, z, t)$ where $m(v) = x^2 + y^2 + z^2 - t^2$ is called the Lorentz metric.

(c) If $u(x, y, z, t)$ is any function in $C^2(\mathbb{R}^4)$ and L is a Lorentz metric, let $U(x, y, z, t) = u(L(x, y, z, t))$. Show that

$$\Delta U - U_{t,t} = \Delta u - u_{t,t}$$

Hence, if u solves the wave equation, then also U solves the wave equation.

(d) By considering the level sets of m explain the meaning of a Lorentz transformation in more geometrical terms.

2. Find all the three-dimensional plane waves; that is, all the solution of the wave equation of the form $u(x, t) = f(\langle k, x\rangle - ct)$ where $k \in \mathbb{R}^3$ is a fixed vector, $c > 0$ and $f \in C^2(\mathbb{R})$.

Problems for practice and discussion

1. Verify that $u(x, t) = (c^2t^2 - x^2 - y^2 - z^2)^{-1}$ satisfies the 3-dimensional wave equation except on the light cone.

2. Prove the principle of causality for two space dimensions.

To Read

1. Section 9.1 and 9.2 in *Partial Differential Equations: An Introduction* by W. Strauss, and Section 6.2 in *Partial Differential Equations* by R. Choksi