MAT351 Partial Differential Equations **Fall, Assignment 13**

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Hand-in Problems (Due till March 8 before lecture, via crowdmark)

1. Let u solve

$$u_{t,t} = \Delta u \text{ in } \mathbb{R}^3 \times (0, \infty)$$
$$u = \phi,\ u_t = \psi \text{ on } \mathbb{R}^3 \times \{0\}$$

   where ϕ and ψ are smooth and have compact support.

   Show there exists a constant $C > 0$ such that

$$u(x,t) \le \frac{C}{t}\ \forall x \in \mathbb{R}^3,\ t > 0.$$

2. The solution of the wave equation $u_{t,t} = c^2 \Delta u$ in $2D$ is given by Poisson's formula. Use Hadamard's method of descent to recover D'Alembert's formula for the solution of the wave equation in space dimension $n = 1$.

Problems for practice and discussion

1. *Maxwell's equations* of electromagnetism in vacuum are

$$\nabla \times E = -\frac{\partial B}{\partial t},\ \nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t},\ \nabla \cdot E = 0,\ \nabla \cdot B = 0$$

   where ϵ_0 is the vacuum permitivity, μ_0 is the vacuum permeability, $\nabla \cdot V$ is the divergence $\operatorname{div} V$ of a vectorfield V and

$$\nabla \times V = \begin{pmatrix} \frac{\partial V_2}{\partial x_3} - \frac{\partial V_3}{\partial x_2} \\ \frac{\partial V_3}{\partial x_1} - \frac{\partial V_1}{\partial x_3} \\ \frac{\partial V_1}{\partial x_2} - \frac{\partial V_2}{\partial x_1} \end{pmatrix}$$

   is the rotation vectorfield of V.

   Show that E_i and B_i, $i = 1, \dots 3$, solve the wave equation $u_{t,t} = \frac{1}{\mu_0 \epsilon_0} \Delta u$.
   Hint: Check first that $\nabla \times (\nabla \times V) = \nabla(\operatorname{div} V) - \operatorname{div} \nabla V$ *where* $V : \mathbb{R}^3 \to \mathbb{R}^3$ *is a smooth vectorfield. The divergence* $\operatorname{div} \nabla V$ *is the vectorfield that is formed of the divergences of the components* $V_i, i = 1, 2, 3,$ *of* V.

2. Derive the conservation of energy for the wave equation on a domain Ω (with smooth boundary) with Dirichlet or Neumann boundary conditions. What about Robin conditions?

To Read

1. Section 9.3, 9.4 and 9.5 in *Partial Differential Equations: An Introduction* by W. Strauss.