MAT351 Partial Differential Equations **Fall, Assignment 14**

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Hand-in Problems (Due till March 17 before lecture, via crowdmark)

1. Prove the converse in Theorem 10.20 (lecture notes). That is, prove that a level surface of $t - g(x) = f(x,t)$ is characterisic if g satisfies the *eikonal equation*

$$|\nabla g| = \frac{1}{c} \text{ on } \mathbb{R}^3.$$

Hint: Differentiate the equation to show that $\sum_{i=1}^{3} g_{i,j} g_j$ $\forall i = 1, \ldots, n$. *Let* $x(t)$ *be a solution of the differential equation* $\frac{d}{dt}x(t) = c^2 \nabla g \circ x(t)$. *Show that* $\frac{d^2}{dt^2}x(t) = 0$. *Show* f *is constant along* $(x(t), t)$. *Conclude that* $S = \{(x,t) \in \mathbb{R}^4 : t - g(x) = 0\}$ *is a characteristic surface.*

2. Proof Theorem 10.25 for one space dimension: Consider $u_{t,t} = c^2 u_{x,x}$ with initial conditions on a surface $S = \{(x,t) \in \mathbb{R}^2 : t = \gamma(x)\}$ for $\gamma : \mathbb{R} \to \mathbb{R}$ smooth, given by

$$u((x, \gamma(x)) = \phi(x), \quad \frac{\partial u}{\partial N} = \psi(x).$$

If S is spacelike, i.e. $\left|\frac{d}{dx}\gamma(x)\right| < \frac{1}{c}$, prove that the initial value problem has a unique solution.

Hint: Recall that the general solution of $u_{t,t} = c^2 u_{x,x}$ *on* $\mathbb{R}^2$ *is given by* $u(x,t) = f(x+ct) + g(x-ct)$ *for* $f, g \in C^2(\mathbb{R})$.

Problems for practice and discussion

1. Show that a solution $u \in C^2(\mathbb{R}^4)$ of $u_{t,t} - c^2 \Delta u = f$ for $f(x,t), (x,t) \in \mathbb{R}^4$, with $u(x,0) = u_t(x,0) = 0$ satisfies

$$u(x,t) = \int_0^t (\mathcal{S}(s-t)f)(x,s)ds$$

where $(\mathcal{S}(t)\phi)(x) = t\bar{\phi}(x,t)$ is the *source operator* ($\bar{\phi}(x,t) = \fint_{\partial B_{ct}} \phi(y)ds(y)$).

Hint: Recall Duhammel's principle.

2. (a) Derive a solution formula for the heat equation

$$u_t = k\Delta u \text{ on } \mathbb{R}^{n+1} \text{ with } u(x,0) = \phi(x) \text{ for } x \in \mathbb{R}^n$$

in the cases where the initial condition ϕ is given by

$$\phi(x_1, \ldots, x_n) = \Pi_{i=1}^n \phi_i(x_i)$$

with $\phi_i, i = 1, \ldots, n$ bounded, continuous functions on $\mathbb{R}$.

(b) Argue that the formula that you derived holds for every bounded continuous function ϕ on $\mathbb{R}^n$.

You may use the fact that there is a sequence of finite sums of products as in part (a) which converge uniformly to ϕ.

To Read

1. Section 10.1 and 10.2 in *Partial Differential Equations: An Introduction* by W. Strauss.