MAT351 Partial Differential Equations **Fall, Assignment 15**

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Hand-in Problems (Due till March 22, 11:59 pm, via crowdmark)

1. (a) Show directly from the differential equation $v'' + (\lambda - x^2)v = 0$ that the functions $H_k(x)e^{-x^2/2}$ are mutually orthogonal on $\mathbb{R}$. That is

$$\int_{-\infty}^{\infty} H_k(x)H_l(x)e^{-x^2}dx = 0 \text{ for } k \neq l,\ k, l \in \mathbb{N}.$$

   (b) Explain how one can derive the Hermite polynomials H_k recursively from the Gram-Schmidt algorithm. Note that you are not asked to perform a computation.

2. (a) Show that if $\lambda \neq 2k + 1$, any solution of Hermite's differential equation is a power series but not a polynomial.

   (b) If a solution of Hermite's differential equation is not a polynomial, show that it cannot satisfy the condition at infinity. (*Hint: Use the recursion formula for the coefficients and compare with the power series of* e^{x^2}.)

Problems for practice and discussion

1. (a) Derive the first four Hermite polynomial directly from the recursion formula for the coefficients.

   (b) Show that all the Hermite polynomials are given by the formula

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$$

   up to a constant factor. (*Hint: Show that* $H_{k+1}(x) = 2xH_k(x) - H_k'(x)$.)

2. Find the solution of the diffusion equation in the half space with Neumann boundary condition. (*Hint: Use the method of reflection.*)

3. About the hydrogen atom: If $\lambda > 0$, why would you expect that Laguerre's differential equation

$$R'' + \frac{2}{r}R' + \frac{2}{r}R + \lambda R = 0$$

   does not have a solution v that satisfies the condition $v(x) \to 0$ for $|x| \to \infty$.

To Read

1. Section 10.3, 10.4, 10.5, 10.6 and 10.7 in *Partial Differential Equations: An Introduction* by W. Strauss.