MAT351 Partial Differential Equations **Fall, Assignment 16**

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Hand-in Problems (Due till April 5, 11:59 pm, via crowdmark)

1. (a) Write Schroedinger's equation $iu_t = -\frac{1}{2}\Delta u + V \cdot u$ in *two* dimensions in polar coordinates where $V = V(r)$, $r^2 = x^2 + y^2$, is a radial potential funcion. Separate the variables to find special solutions of the form $u(t, r, \theta) = T(t) \cdot R(r) \cdot \Theta(\theta)$.

   (b) Assume $V(r) = \frac{1}{2}r^2$. Substitute $\rho = r^2$ and $R(r) = e^{-\rho/2}\rho^{-n/2}L(\rho)$ to show that L satisfies Laguerre's differential equation

$$L'' + \left(\frac{\nu+1}{\rho} - 1\right)L' + \frac{\mu}{\rho}L = 0$$

   for some constants μ and ν. Explain how to obtain polynomial solutions.

2. (a) Let $f(x)$ be a function such that

$$f(0) = f(3) = 0, \quad \int_0^3 [f(x)]^2 dx = 1 \quad \text{and} \quad \int_0^3 [f'(x)]^2 dx = 1.$$

   Find such a function if you can. If it cannot be found, explain why not?

   Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$.

   (b) Show that the smallest Neumann eigenvalue for $-\Delta$ on Ω is given by $\tilde{\lambda}_1 = 0$. What are the corresponding eigenfunction.

   (c) If Ω is connected, show that $\tilde{\lambda}_1$ is simple, that is the space of eigenfunctions is 1-dimensional. To show this argue that $\tilde{\lambda}_2$ is strictly positive. You can assume without proof that smooth eigenfunctions exist.

Problems for practice and discussion

1. Derive the recursion relations

$$J_{s\pm1}(z) = \frac{s}{z}J_s(z) \mp J_s'(z) \quad \text{and} \quad J_{s-1}(z) + J_{s+1}(z) = \frac{2s}{z}J_s(z)$$

   for the Bessel functions $J_s(z)$, $s \in \mathbb{R}\backslash\{-n : n \in \mathbb{N}\}$.

2. Estimate the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in the triangle

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x + y < 1, x > 0, y > 0\}$$

   using the Rayleigh quotient with trial function $xy(1 - x - y)$.

3. Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary. Denote by λ_n, $n \in \mathbb{N}$, the sequence of eigenvalues of the negative Dirichlet Laplace operator, $-\Delta$ on $\mathcal{E}_0 = \{w \in C^2(\overline{\Omega}) : w|_{\partial\Omega} = 0\}$, and by v_n, $n \in \mathbb{N}$, an orthonormal set of corresponding eigenfunctions.

   Assume $u \in C^2(\overline{\Omega} \times [0, \infty))$ solves the heat equation $u_t = \Delta u$ with homogeneous Dirichlet boundary conditions and initial value $u(\cdot, 0) = f$ on Ω. Express u in terms of the Dirichlet eigenvalues and the eigenfunctions given above.

To Read

1. Section 11.1, 11.2 and 11.3 in *Partial Differential Equations: An Introduction* by W. Strauss, Section 16.4 in *Partial Differential Equations* by R. Choksi.