MAT351 Partial Differential Equations **Fall, Assignment 17**

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Hand-in Problems (Due till April 12, 11:59 pm, via crowdmark)

1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega$ smooth. Let N be the unit normal vector field along Ω. Let $u \in C^2(\overline{\Omega})$ be a minimizer of

$$\{w \in C^1(\overline{\Omega}) \text{ with } \int_\Omega w dx = 0\} \ni w \mapsto R(w) = \frac{E(w)}{\|w\|_2^2}$$

where $E(w) = \int_\Omega |\nabla w|^2 dx$ and $\|w\|_2^2 = \int_\Omega w^2 dx$. Show that u solves

$$-\Delta u = \lambda u \text{ on } \Omega$$
$$\frac{\partial u}{\partial N} = 0 \text{ on } \partial\Omega$$

with $\lambda = R(u) > 0$.

Hint: First consider $u + \epsilon\phi$ with $\phi \in \mathcal{E}_0$.

Problems for practice and discussion

1. Estimate the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions in the triangle

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x + y < 1, x > 0, y > 0\}$$

using the Rayleigh quotient of the trial function $xy(1 - x - y)$.

2. For $-\Delta$ on the interior $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2/4 < 1\}$ of an ellipse with Dirichlet boundary conditions use the monotonicity of the eigenvalues w.r.t. the domain to find estimates for the first two eigenvalues. More precisely, find upper and lower bounds.
Hint: Inscribe or circumscribe rectangles or circles, for which we know the exact values.

3. For the eigenvalue problem $-u'' = \lambda u$ in the interval $(0, 1)$ with $u(0) = u(1) = 0$ choose the pair of trial functions $x - x^2$ and $x^2 - x^3$ and compute the Rayleigh Ritz approximation for the first two eigenvalues and compare with the exact values.

To Read

1. Section 11.6, 12.1, 12.2, 12.3 and 12.4 in *Partial Differential Equations: An Introduction* by W. Strauss, Section 16.4 in *Partial Differential Equations* by R. Choksi.