MAT351 Partial Differential Equations **Fall, Assignment 3**

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Hand-in Problems (Due till October 19 before lecture, via crowdmark)

H1 Use the graphical method to sketch the solution of **Burgers' equation** $u_t + uu_x = 0$ with initial values

$$u(x,0) = \begin{cases} 1 & \text{if } x < -1, \\ 0 & \text{if } -1 \le x \le 0, \\ 2 & \text{if } 0 < x < 1, \\ 0 & \text{if } x > 1 \end{cases}$$

that satisfies both the Rankine-Hugoniot jump condition and the Lax entropy condition. Indicate the location of the shocks and rarefaction waves.

H2 Reduce the elliptic equation

$$u_{x,x} + 3u_{y,y} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{x,x} + v_{y,y} + cv = 0$ by a change of the dependent variable of the form $u = ve^{\alpha x + \beta y}$ and then a change of scale of the form $y' = \gamma y$.

Problems for discussion

1. Consider the conservation law

$$u_t + (f(u))_x = 0$$

where f is a non-decreasing function on the real line with $A(0) = 0$.

(a) If this is a model for traffic flow with $u(x,t) \ge 0$ denoting the density of cars, how would you interpret $f'(u(x,t))$? Argue that the function $f(x)$ should be concave?

(b) Let f be as above, and let u be an entropy solution. Interpret the formation of shocks, the Rankine-Hugoniot condition, and the Lax' entropy condition in this traffic model.

2. (a) What is the type of each of the following equations?
   - $u_{x,x} - u_{x,y} + 2u_y + u_{y,y} - 3u_{y,x} + 4u = 0$,
   - $9u_{x,x} + 6u_{x,y} + u_{y,y} + u_x = 0$.

   (b) Find the regions in the xy plane where the equation

$$(1 + x)u_{x,x} + 2xyu_{x,y} - y^2 u_{y,y} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch these regions.

To Read

1. Section 1.3, 1.6 and 2.1 in *Partial Differential Equations: An Introduction* by W. Strauss.