MAT351 Partial Differential Equations **Fall, Assignment 5**

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Hand-in Problems (Due till October 26 before lecture, via crowdmark)

H1 Prove the *comparison principle* for the diffusion equation: Let u and v be two C^2 solutions of $u_t = ku_{x,x}$ with $k > 0$ on $[0, l] \times [0, \infty)$. If $u(x,t) \le v(x,t)$ for $(x,t) \in [0,l] \times \{0\}$, for $(x,t) \in \{0\} \times [0,\infty)$ and for $(x,t) \in \{l\} \times [0,\infty)$, then $u \le v$ on $[0,l] \times [0,\infty)$.

H2 Consider the diffusion equation $u_t = ku_{x,x}$ for $k > 0$ on $[-1, 1]$ with *Robin boundary conditions*

$$u_x(-1,t) - au(-1,t) = 0 = u_x(1,t) + au(1,t).$$

(a) If $a > 0$ show that the energy $E(t) = \int_{-1}^{1} [u(x,t)]^2 \, dx$ is decreasing.

(b) If a is much smaller than 0, show by example that energy may increase or decrease. *Hint: Consider solutions of the form* $h(t)\cosh(bx)$.

Problems for discussion

1. Proof the properties $(a), (b), (c), (d)$ and (e) from the lecture.

2. (a) Compute $\int_{-\infty}^{\infty} e^{-x^2} dx$.

   Hint: Use polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$ *and the transformation formula to transform the double integral* $\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$ *into an integral that we can calculate.*

   (b) Use (a) to show that $\int_{-\infty}^{\infty} e^{-r^2} dr = \sqrt{\pi}$. Use the substitute $p = \frac{x}{\sqrt{4kt}}$ to show

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{x}{\sqrt{4kt}}\right)^2} dx = 1.$$

3. Solve the diffusion equation $u_t = ku_{x,x}$ on $\mathbb{R}$ with the initial condition $u(x,0) = x^2$ by the following special method: Given the solution u show that $u_{x,x,x}$ satisfies the diffusion equation on $\mathbb{R}$ with $u(x,0) = 0$. By uniqueness of the solution $u_{x,x,x} = 0$. Integrating trice yields $u(x,t) = A(t)x^2 + B(t)x + C(t)$. Determine A, B and C.

4. (a) Solve the previous problem using the fundamental solution. By uniqueness the resulting formula must agree with the previous answer.

   (b) Deduce the value of

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx.$$

To Read

1. Section 2.4, 2.5, 3.1 and 3.2 in *Partial Differential Equations: An Introduction* by W. Strauss.