MAT351 Partial Differential Equations **Fall, Assignment 7**
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Hand-in Problems (Due till November 23 before lecture, via crowdmark)

1. Consider the Laplace equation in polar coordinates

$$u_{r,r} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta,\theta} = 0,\ (r,\theta) \in (0,\infty) \times \mathbb{R},$$

   with periodic boundary conditions in $\theta \in \mathbb{R}$: $u(r\theta + 2\pi) = u(r,\theta)$ for all $\theta \in \mathbb{R}$.

   (a) Set $u(r,\theta) = f(r)g(\theta)$ and separate variables to obtain a pair of ODEs for f ang g.

   (b) Solve these ODEs to obtain special solutions for the Laplace equation. (*Hint: Try* $f(r) = r^\alpha$.)

2. Suppose that ϕ is bounded and continuous everywhere except for a jump discontinuity at $a \in \mathbb{R}$, i.e. the right- and left-sided limits $\phi(a^+)$ and $\phi(a^-)$ exist where

$$\phi(x^+) = \lim_{y \downarrow x} \phi(y) \quad \& \quad \phi(a^-) = \lim_{y \uparrow x} \phi(y)$$

   exist. Let S be the fundamental solution of the equation $u_t = ku_{x,x}$ and set

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$

   (a) Explain why $u(x,t)$ solves $u_t = ku_{x,x}$ on $\mathbb{R} \times (0,\infty)$.

   (b) Show that $\lim_{t\downarrow 0} u(x,t) = \frac{1}{2}\left(\phi(x^+) + \phi(x^-)\right)$ for all $x \in \mathbb{R}$.
   Hint: Change variables and show that

$$\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{z^2}{4}} \phi(\sqrt{4kt}z)dz \to \frac{1}{2}\phi(0^+) \text{ for } t \downarrow 0.$$

Problems for discussion

1. A quantum-mechanical particle on the line $\mathbb{R}$ with an inifinite potential outside the interval $(0,l)$ ("particle in a box") is given by Schroedinger's equation $u_t = iu_{x,x}$ on $(0,l)$ with Dirichlet conditions at the ends. Here, i is the imaginary unit, satisfying $i^2 = -1$.

   Apply the separation-of-variables-method to this problem as we did in the lecture to find solutions to this problem.

To Read

1. Section 4.1, 4.2 and 5.1 in *Partial Differential Equations: An Introduction* by W. Strauss.