MAT351 Partial Differential Equations **Fall, Assignment 8**

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Hand-in Problems (Due till December 2 before lecture, via crowdmark)

1. Given a function ϕ on the interval $(0, l)$ the even extension of ϕ is defined as

$$\phi_{even}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < l \\ \phi(-x) & \text{for } -l < x < 0. \end{cases}$$

The even extension is not necessarily defined at 0. The odd extension is

$$\phi_{odd}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < l \\ -\phi(-x) & \text{for } -l < x < 0 \\ 0 & \text{for } x = 0. \end{cases}$$

(a) Show that

$$\int_{-l}^{l} \phi_{odd}(x)dx = 0 \quad \text{and} \quad \int_{-l}^{l} \phi_{even}(x)dx = 2\int_{0}^{l} \phi(x)dx.$$

(b) Let ϕ be an odd function on $(-l, l)$. Use (a) to show that the full Fourier series on $(-l, l)$ has only sine terms. If $\phi(x)$ is an even function, show that the full Fourier series on $(-l, l)$ has only cosine terms.

(c) Show that the Fourier sine and the Fourier cosine series of ϕ defined on $(0, l)$ can be derived from the full Fourier series on $(-l, l)$.

2. (a) Show that the mixed boundary condition $X(0) = 0$ and $X_x(l) = 0$ for the operator $-\frac{d^2}{dx^2}$ lead to the following set of eigenfunctions $\sin\left(\frac{(2n+1)\pi}{2l}x\right)$, $n \in \mathbb{N} \cup \{0\}$.

(b) If $\phi(x)$ is a function on $(0, l)$, derive the expansion

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{(2n+1)\pi}{2l}x\right), \; x \in (0, l).$$

by the following method. Extend $\phi(x)$ to the function $\tilde{\phi}(x)$ defined by $\phi(x) = \tilde{\phi}(x)$ for $0 \le x \le l$ and $\tilde{\phi}(x) = \phi(2l - x)$ for $l \le x \le 2l$ (that means we are extending evenly across $x = l$). Write the Fourier sine serie for $\tilde{\phi}(x)$ on the interval $(0, 2l)$ and the corresponding coefficients.

(c) Show that every second term vanishes.

(d) Rewrite the formula for C_n as an integral of the original function ϕ on the interval $(0, l)$.

Problems for discussion

1. (*Gram-Schmidt orthogonalization procedure*) If $X_1, X_2, \dots$ is any sequence (finite or infinite) of linearly independent vectors in any vector space with an inner product, it can be replaced by a sequence of linear combinations that are mutually orthogonal. The idea is that a each step one substracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 := X_1 / \|X_1\|$. Second, we define

$$Y_2 := X_2 - (X_2, Z_1)Z_1 \quad \text{and} \quad Z_2 = \frac{Y_2}{\|Y_2\|}$$

Third, we define

$$Y_3 = X_3 - (X_3, Z_2)Z_2 - (X_3, Z_1)Z_1 \quad \text{and} \ Z_3 = \frac{Y_3}{\|Y_3\|},$$

and so on.

   (a) Show that all the vectors $Z_1, Z_2, \dots$ are orthogonal to each other.

   (b) Apply the Gram-Schmidt procedure to the pair of functions $\cos x + \cos(2x)$ and $3\cos(x) - 4\cos(2x)$ in the interval $(0, \pi)$ to get an orthogonal pair.

To Read

1. Section 5.2, 5.3, 5.4 and 5.5 in *Partial Differential Equations: An Introduction* by W. Strauss.