MAT351 Partial Differential Equations
Lecture 3

September 21, 2020

Last Lecture

Theorem (Fundamental Theorem of Calculus of Variations, 1st version)

Consider $f \in C^0(\mathbb{R}^n)$. *If*

$$\int f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = 0 \ \ \forall \varphi \in C_c^0(\mathbb{R}^n), \ \varphi \geq 0 \implies f \equiv 0.$$

Theorem (Fundamental Theorem of Calculus of Variations, 2nd version)

Consider $f \in C^0(\mathbb{R}^n)$. *If*

$$\int_\Omega f(\mathbf{x})d\mathbf{x} = \int f(\mathbf{x})1_\Omega(\mathbf{x})d\mathbf{x} = 0 \ \ \forall \Omega \ \ \textit{domain with smooth boundary in } \mathbb{R}^n$$

$\implies f \equiv 0.$ *A domain is connected and open subset of* $\mathbb{R}^n$.

$$1_\Omega(\mathbf{x}) = \begin{cases} 1 & \textit{if } \mathbf{x} \in \Omega \\ 0 & \textit{if } \mathbf{x} \notin \Omega. \end{cases} \ \ \textit{is the indicator or characteristic function of } \Omega \subset \mathbb{R}^n.$$

Hence, if $f, g \in C^0(\mathbb{R}^n)$ and

$$\int_\Omega g(\mathbf{x})d\mathbf{x} = \int_\Omega f(\mathbf{x})d\mathbf{x} \ \forall \Omega \subset \mathbb{R}^n \text{ with smooth boundary } \Rightarrow f \equiv g.$$

We derived 3 types of PDE:

- **Simple Transport Equation**:

$$u_t + V \cdot \nabla_{\mathbf{x}} u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{1}$$

- **Scalar Conversation Laws**:

$$u_t + \nabla_{\mathbf{x}} \mathbf{f}(u) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{2}$$

- **Diffusion Equation**:

$$u_t = -\lambda \Delta_{\mathbf{x}} u = -\nabla \cdot (\lambda \nabla_x u) \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{3}$$

  This equation is also known as **heat equation** since it also describes *heat flow*.
  In this context the parameter λ describes the heat conductivity, that also can depend on $\mathbf{x}$.
  For simplicity, we will set $\lambda = 1$ in the following.

If u does not change in t, the left hand side is 0. The equation becomes

- **Laplace Equation**

$$0 = \Delta u \quad \text{in } \mathbb{R}^n.$$

  Solutions of this equation are called harmonic functions.

General Solution

A PDE

$$F(\mathbf{x}, u, Du, D^2u) = g(\mathbf{x}),\ \mathbf{x} \in \Omega \tag{4}$$

can have infinitely many solutions.

The general solution is the collection of all u that are sufficiently smooth and satisfy (4).

Example

What is the general solution of the following PDE?

$$au_x + bu_y = 0,\ \text{in } \mathbb{R}^2, a \neq b. \tag{5}$$

Rewrite (5) as $(a, b) \cdot \nabla u = 0 = \frac{\partial u}{\partial V}$ where $V = (a, b) \in \mathbb{R}^2$.

Hence, a solution u must be constant in direction V.

The lines parallel to V (*Characteristics*) have the form

$$y = \tfrac{b}{a}x + \tfrac{c}{a} \text{ or } bx - ay = c \text{ where } c \in \mathbb{R} \text{ is a parameter.}$$

Therefore, the solution u depends only on $bx - ay$. The general solution is

$$u(x, y) = f(bx - ay) \ \text{ for } \ f \in C^1(\mathbb{R}).$$

We can solve the previous PDE also using a coordinate change.

Define *new coordinates* via

$$x'(x,y) = ax + by, \quad y'(x,y) = bx - ay.$$

The chain rule yields

$$u_x = u_{x'} \frac{\partial x'}{\partial x} + u_{y'} \frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$

and similar $u_y = au_{x'} - bu_{y'}$. Hence

$$0 = au_x + bu_y = (a^2 + b^2)u_{x'} \ \Rightarrow \ u_{x'} = 0 \ \text{ since } a^2 + b^2 \neq 0.$$

The general solution for this PDE is $f(y') = f(bx - ay)$ for $f \in C^1(\mathbb{R})$.

Auxiliary Conditions

We want to specify an **Auxiliary Condition** that eventually yields a unique solution.

For instance, for

$$au_x + bu_y = 0 \text{ in } \mathbb{R}^2, a \neq b$$

we could set the condition that $u(0, y) = g(y)$ for $g \in C^1(\mathbb{R})$.
The general solution is $f(bx - ay) = u(x, y)$ for $f \in C^1(\mathbb{R})$.
Let us determine f such that u satisfies the previous auxiliary condition:

$$g(y) = u(0, y) = f(-ay).$$

Hence, the solution is $u(x, y) = g(-\frac{1}{a}(bx - ay)) = g(y - \frac{b}{a}x)$.

Remark: Roughly, if a PDE has n independent variable (in a domain Ω) an auxiliary condition is a set of specified values on an $n - 1$ dimensional subset of Ω.

Initial Value Conditions

Let $\phi \in C^2(\mathbb{R}^n)$.

We can search for solutions $u(\mathbf{x}, t)$ of the diffusion equation

$$u_t + \Delta u = 0 \quad \text{on } \mathbb{R}^n \times [t_0, \infty) \tag{6}$$

such that $u(\mathbf{x}, t_0) = \phi(\mathbf{x})$ for some t_0.

Definition (Initial Value Problem (IVP) for the Diffusion Equation)

Let $\phi \in C^2(\mathbb{R}^n)$. The IVP for the diffusion equation is (6) together with

$$u(\mathbf{x}, 0) = \phi(\mathbf{x}) \ \text{ for } \mathbf{x} \in \mathbb{R}^n. \tag{7}$$

Definition (IVP for the Wave Equation)

The PDE $u_{t,t} = c^2 \Delta u$ on $\mathbb{R}^n \times [0, \infty)$ is called *Wave equation*. The corresponding IVP is

$$u_{t,t} = c^2 \Delta u \ \text{ on } \mathbb{R}^n \times [0, \infty),$$

$$u(\mathbf{x}, 0) = \phi(\mathbf{x}) \ \text{ and } u_t(\mathbf{x}, 0) = \psi(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{R}^n.$$

Here c is a physical parameter that describes the speed of the wave.

We will study the wave equations later in more detail.

Boundary Value Conditions

Let Ω be a domain with smooth boundary. Consider the Laplace equation

$$\Delta u = 0 \quad \text{on } \Omega.$$

The domain $\Omega \subset \mathbb{R}^n$ depends the (physical or mathematical) background.

Definition

There are 3 common types of boundary conditions for the Laplace equation: Let $\phi, \psi \in C^0(\partial\Omega)$.

- **Dirichlet boundary conditions (DC)**:

$$u(\mathbf{x}) = \phi(\mathbf{x}) \quad \text{on} \quad \partial\Omega.$$

- **Neumann boundary conditions (NC)**:

$$\frac{\partial u}{\partial N}(\mathbf{x}) = \phi \quad \text{on} \quad \partial\Omega$$

  where $\frac{\partial u}{\partial N} = \nabla u \cdot N$ is the derivative of u in direction of N, and N is the outer unit normal vector field of Ω.

- **Robin boundary conditions (RC)**:

$$\frac{\partial u}{\partial N}(\mathbf{x}) + \psi(\mathbf{x})u(\mathbf{x}) = \phi(\mathbf{x})$$

If $\phi \equiv 0$ we call the corresponding boundary condition homogeneous, otherwise inhomogeneous.

Example

The problem $\Delta u = 0$ on $B_1(0) = \{|\mathbf{x}|_2 \leq 1\}$ has infinitely many solutions $u \equiv c$, $c \in \mathbb{R}$.

Imposing a homogeneous DC picks the solution $u \equiv 0$.

On the other hand, any solution $u \equiv c$, $c \in \mathbb{R}$, satisfies the homogeneous NC. Hence, this problem is not *wellposed*.

We also can consider a general linear PDE of order 2:

$$\sum_{i,j=1}^{n} a_{i,j} u_{x_i,x_j} + \sum_{k=1}^{n} b_k u_{x_k} + cu = 0 \text{ on } \Omega$$

together with Dirichlet, Neumann or Robin boundary conditions.

Example (Eigenvalue equation (in 1D))

Consider

$$\frac{d}{dx} u + u = 0 \quad \text{on} \quad [0, \pi] \subset \mathbb{R}.$$

A solution for the homogenenous Dirichlet problem (DP) is given by

$$u(x) = c \sin x, \ \forall c \in \mathbb{R},$$

a solution for the homogeneous Neumann problem (NP) is given by

$$u(x) = c \cos x, \ \forall c \in \mathbb{R}.$$

Global Condition/Condition at Infinity

One also can assume a global condition like

$$\int_\Omega u(\mathbf{x})d\mathbf{x} = 0 \quad \text{or} \quad \int |u(\mathbf{x})|^2 d\mathbf{x} = c \text{ for } c > 0 \text{ fixed.}$$

Example

The unique solution of the Neumann or Dirichlet eigenvalue problem such that

$$\int_0^\pi |u(x)|^2 dx = \pi$$

is given by $u(x) = \cos x$ and $u(x) = \sin x$, respectively.

If Ω is unbounded (like $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^{n-1} \times [0,1]$) one might also impose such an integral condition.
This forces u to vanish for $|\mathbf{x}| \to \infty$.

Or we have directly a boundary condition of the form

$$\lim_{|\mathbf{x}| \to 0} u(\mathbf{x}) = 0 \quad \text{or} \quad \lim_{|\mathbf{x}| \to 0} \frac{u(\mathbf{x})}{|\mathbf{x}|^\alpha} = 0 \quad \text{for } \alpha > 0.$$

Wellposed Problems

Definition

Consider a PDE of order k

$$F(\mathbf{x}, u, Du, \dots, D^k u) = g(\mathbf{x})$$

on a domain $\Omega \subset \mathbb{R}^n$ with a certain auxiliary condition.
The corresponding problem *wellposed* if it satisfies the following fundamental properties.

1. **Existence**: There exists a solution.
2. **Uniqueness**: The solution is unique.
3. **Stability**: The unique solution depends in a stable manner on the data of the problem. This means if we change the data or the auxiliary conditions a little bit then also the unique solution only changes a little bit.

Remark

Since we often can measure data only up to certain degree of precision, stability guarantees that a solution of our problem that relies on the given data which approximates the exact data, is close to the solution that we would deduce from the exact data.

Introduction to the Method of Characteristics

We found the general solution of $au_x + bu_y = 0$. Solutions are constant on lines parallel to (a, b).

Now, we consider

$$u_x + yu_y = 0 \Longleftrightarrow (1, y) \cdot \nabla u = 0 \text{ in } \mathbb{R}^2.$$

Instead of straight lines, now we looking for curves $(x, y(x))$ such that

$$\frac{d}{dx}(x, y(x)) = (1, y) \Leftrightarrow \frac{dy}{dx} = y$$

Hence $y(x) = Ce^x$, and a solution u satisfies

$$\frac{d}{dx}u(x, y(x)) = \nabla u \cdot (1, y) = 0$$

and

$$u(x, y(x)) = u(0, y(0)) = u(0, C)$$

is independent of x.

Let $g \in C^1(\mathbb{R})$.
Since for every tuple (x, y) there exists a unique $C(x, y)$ such that $(x, y) = (x, C(x, y)e^x)$.
Then $u(x, y) := u(0, C(x, y)) = g(ye^{-x})$ satisfies

$$u_x + yu_y = g'(ye^{-x})(-ye^{-x}) + yg'(ye^{-x})e^{-x} = 0.$$

Therefore $u(x, y) = g(ye^{-x})$ solves the PDE with auxiliary condition $g(y) = u(0, y)$.