MAT351 Partial Differential Equations
Lecture 4

September 23, 2020

Introduction to the Method of Characteristics

Last Lecture:

We found the general solution of $au_x + bu_y = 0$. Solutions are constant on lines parallel to (a, b).

As another example we consider

$$u_x + yu_y = 0 \Longleftrightarrow (1, y) \cdot \nabla u = 0 \ \text{ in } \mathbb{R}^2$$

with the auxiliary condition $u(0, y) = g(y)$ for $g \in C^1(\mathbb{R})$.

Instead of straight lines, now we looking for curves $(x, y(x))$ such that

$$\frac{d}{dx}(x, y(x)) = (1, y) \ \Leftrightarrow \ \frac{d}{dx}x = 1 \ \& \ \frac{d}{dx}y = y$$

Hence $y(x) = y_0 e^x$ with $y(0) = y_0$.

Moreover, a solution u of the PDE satisfies along the curve $(x, y(x))$:

$$\frac{d}{dx}u(x, y(x)) = \nabla u \cdot (1, y) = u_x + yu_y = 0$$

and $u(x, y(x)) = u(0, y(0)) = u(0, y_0)$ is independent of x.

Given a point $(\hat{x}, \hat{y}) \in \mathbb{R}^2$ we want to find y_0 and $y(\cdot)$ with $y(0) = y_0$ and $y(\hat{x}) = \hat{y}$.

Then, we know the value of u in $(\hat{x}, \hat{y})$: It is

$$u(\hat{x}, \hat{y}) = u(\hat{x}, y(\hat{x})) = u(0, y_0) = g(y_0).$$

But by the formula for $y(x)$, we can indeed find such y_0: it is $y_0 = \hat{y}e^{-\hat{x}}$. Therefore we can write

$$u(\hat{x}, \hat{y}) = g(\hat{y}e^{-\hat{x}}).$$

This u satisfies the PDE with the given auxiliary condition. (let us drop $\hat{\cdot}$) Indeed

$$u_x + yu_y = g'(ye^{-x})(-ye^{-x}) + yg'(ye^{-x})e^{-x} = 0.$$

Method of characteristics for linear equations

We consider a general linear PDE of order 1 with 2 independent variables x, y:

$$a(x,y)u_x + b(x,y)u_y = c_1(x,y)u + c_2(x,y) \text{ in } \mathbb{R}^2 \tag{1}$$

where $a, b, c_1, c_2 \in C^2(\mathbb{R}^2)$. Assume there is an auxiliary condition given as follows.

$$u(x,y) = u_0(x,y) \text{ for } (x,y) \in \Gamma$$

where Γ is a 1 dimensional subset in $\mathbb{R}^2$ and $u_0 : \Gamma \to \mathbb{R}$.

We can write (1) also as equation for directional derivatives:

$$V(x,y) \cdot \nabla u = c_1(x,y)u + c_2(x,y)$$

for a vectorfield $(x,y) \mapsto V(x,y) = (a(x,y), b(x,y)) \in \mathbb{R}^2$.

More generally, consider a general linear PDE of order 1 with n independent variables:

$$\sum_{i=1}^{n} a_i(\mathbf{x})u_{x_i} = c_1(\mathbf{x})u + c_2(\mathbf{x}) \text{ in } \mathbb{R}^n. \tag{2}$$

where $a_i, c_1, c_2 \in C^2(\mathbb{R}^n)$ with an auxiliary condition

$$u(\mathbf{x}) = u_0(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma$$

where Γ is a $n-1$ dimensional subset in $\mathbb{R}^n$.

Again we can write (2) as

$$V(\mathbf{x}) \cdot \nabla u = c(\mathbf{x})u + d(\mathbf{x}) \text{ for the vectorfield } V(\mathbf{x}) = (a_1(\mathbf{x}), \dots, a_n(\mathbf{x})).$$

Method of characteristics for linear equations, continued

We want to find the flow curves of V. That is, we have to solve the following ODE:

$$\frac{d}{dt}\gamma_{\mathbf{x}_0}(t) = \dot{\gamma}_{\mathbf{x}_0}(t) = V(\mathbf{x}), \quad \gamma_{\mathbf{x}_0}(0) = \mathbf{x}_0.$$

Recall that

$$\gamma_{\mathbf{x}}(t) = \begin{pmatrix} \gamma_{\mathbf{x}}^1(t) \\ \dots \\ \gamma_{\mathbf{x}}^n(t) \end{pmatrix}, \quad \dot{\gamma}_{\mathbf{x}}(t) = \frac{d}{dt}\gamma_{\mathbf{x}}(t) = \begin{pmatrix} \frac{d}{dt}\gamma_{\mathbf{x}}^1(t) \\ \dots \\ \frac{d}{dt}\gamma_{\mathbf{x}}^n(t) \end{pmatrix}.$$

Assume we have a solution $u \in C^1(\mathbb{R}^n)$. How does u evolve along a flow curve $\gamma_{\mathbf{x}_0}$? The chain rule yields

$$\frac{d}{dt} u \circ \gamma_{\mathbf{x}_0}(t) = \nabla u(\gamma_{\mathbf{x}_0}(t)) \cdot \dot{\gamma}_{\mathbf{x}_0}(t) = \nabla u(\gamma_{\mathbf{x}_0}(t)) \cdot V(\gamma_{\mathbf{x}_0}(t)).$$

On the other hand, by the PDE we have

$$\nabla u(\gamma_{\mathbf{x}_0}(t)) \cdot V(\gamma_{\mathbf{x}_0}(t)) = c_1(\gamma_{\mathbf{x}_0}(t))u(\gamma_{\mathbf{x}_0}(t)) + c_2(\gamma_{\mathbf{x}_0}(t)).$$

This gives us an ODE for the composition $u \circ \gamma_{\mathbf{x}_0}(t) =: z_{\mathbf{x}_0}(t)$:

$$\frac{d}{dt} z_{\mathbf{x}_0}(t) = \dot{z}_{\mathbf{x}_0}(t) = c_1(\gamma_{\mathbf{x}_0}(t)) z_{\mathbf{x}_0}(t) + c_2(\gamma_{\mathbf{x}_0}(t)), \; z_{\mathbf{x}_0}(0) = u_0(\gamma_{\mathbf{x}_0}(0)).$$

Characteristics equations

Definition

Consider linear PDE of order 1 with n independent variables in the form

$$V(\mathbf{x})\nabla u = c_1(\mathbf{x})u + c_2(\mathbf{x}) \quad \text{in } \mathbb{R}^n \quad \text{and } u = u_0 \quad \text{on } \Gamma$$

where Γ is an $n-1$ dimensional subset.

The corresponding *characteristics equations* are

$$\begin{cases} \dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t) & \gamma_{\mathbf{x}_0}(0) = \mathbf{x}_0, \\ \dot{z}_{\mathbf{x}_0}(t) = c_1(\gamma_{\mathbf{x}_0}(t)) z_{\mathbf{x}_0}(t) + c_2(\gamma_{\mathbf{x}_0}(t)) & z_{\mathbf{x}_0}(0) = u_0(\mathbf{x}_0). \end{cases}$$

For the case of 2 independent variable the systems of equations becomes

$$\begin{cases} \dot{x}(t) = a(x(t), y(t)), & x(0) = x_0, \\ \dot{y}(t) = b(x(t), y(t)), & y(0) = y_0, \\ \dot{z}(t) = c_1(x(t), y(t)) z(t) + c_2(x(t), y(t)), & z(0) = u_0(x_0, y_0). \end{cases}$$

Question

How we obtain a solution for the PDE?

Solving the PDE

We will determine the value $u(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ using the method of characteristics.

From the previous consideration, we know that the solution for the PDE (2) must obey the characteristics equations.

Our strategy is:
For $\mathbf{x} \in \mathbb{R}^n$ arbitrary we pick a characteristic $\gamma_{\mathbf{x}_0}$ for an initial point $\mathbf{x}_0 \in \Gamma$ such that $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$ for some $t_0 \in [0, \infty)$.
Then we can solve the characteristic equation for $z_{\mathbf{x}_0} = u \circ \gamma_{\mathbf{x}_0}$

$$\dot{z}_{\mathbf{x}_0}(t) = c_1 \circ \gamma_{\mathbf{x}_0}(t) z_{\mathbf{x}_0}(t) + c_2 \circ \gamma_{\mathbf{x}_0}(t) \;\text{ with } z_{\mathbf{x}_0}(0) = u_0(\mathbf{x}_0)$$

and set

$$z_{\mathbf{x}_0}(t_0) = u \circ \gamma_{\mathbf{x}_0}(t_0) = u(x).$$

Hence, if for every $\mathbf{x} \in \mathbb{R}^n$ we can find a unique $\mathbf{x}_0 \in \Gamma$ and $t_0 \geq 0$ such that

$$\gamma_{\mathbf{x}_0}(t_0) =: \Phi(\mathbf{x}_0, t_0) = \mathbf{x}$$

we can define a function $u : \mathbb{R}^n \to \mathbb{R}$ that will solve the PDE and satisfies the auxiliary condition (by construction). Indeed we have

Proposition

Assuming the function $u : \mathbb{R}^n \to \mathbb{R}$ defined above is a C^1 function, then it solves

$$\nabla u \cdot V(\mathbf{x}) = c_1(\mathbf{x}) u + c_2(\mathbf{x}) \;\text{ in }\; \mathbb{R}^n \;\text{ with }\; u(\mathbf{x}) = u_0(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma.$$

Proof: Let $\mathbf{x} \in \mathbb{R}^n$, assume $\mathbf{x}_0$ is the unique point such that: $\gamma_{\mathbf{x}_0}$ solves $\dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t)$ with $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$.
Then $u(\mathbf{x})$ is defined as $z_{\mathbf{x}_0}(t_0)$.

First, if $x \in \Gamma$, we pick $\mathbf{x}_0 = \mathbf{x}$, $\gamma_{\mathbf{x}_0} = \gamma_{\mathbf{x}}$ and $t_0 = 0$. Then $u(\mathbf{x}) = z_{\mathbf{x}}(0) = u_0(\mathbf{x})$.

For general $\mathbf{x} \in \mathbb{R}^n$ we compute

$$\nabla u(\mathbf{x}) \cdot V(\mathbf{x}) = \nabla u(\gamma_{\mathbf{x}_0}(t_0)) \cdot \dot{\gamma}_{\mathbf{x}_0}(t_0) = \frac{d}{dt} u \circ \gamma_{\mathbf{x}_0}(t_0).$$

Since $z_{\mathbf{x}_0}$ solves the last characteristic equation, the right hand side is equal to

$$c_1(\gamma_{\mathbf{x}_0}(t_0)) u \circ \gamma_{\mathbf{x}_0}(t_0) + c_2(\gamma_{\mathbf{x}_0}(t_0)) = c_1(\mathbf{x}) u(\mathbf{x}) + c_2(\mathbf{x}).$$

Hence u indeed solves the equation. □

Remark

Let us summarize what we assumed here

- We need that any flow curve meets Γ in exactly one point.

  For any $\mathbf{x}$ there exists a unique flow curve $\gamma_{\mathbf{x}_0}$ such that $x_0 \in \Gamma$ and $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$.

  Then, we can solve the initial value problem for $z_{\mathbf{x}_0}$ because the initial value is given by $u(\mathbf{x}_0) = u_0(\mathbf{x}_0)$.

  In other words, we have to solve the equation

$$\Phi_{t_0}|_{\Gamma}(\mathbf{x}_0) = \mathbf{x}$$

  where $\Phi_t(\mathbf{y}) = \gamma_{\mathbf{y}}(t)$ is the flow map of V, and $\Phi_t|_{\Gamma}$ is the restriction of Φ_t to Γ.
- We need that $u \in C^1(\mathbb{R}^n)$.

Example

We find the solution for

$$xu_x + 2u_y = 3u \text{ in } \mathbb{R}^2,\ u(x,0) = \sin x,\ \Gamma = \mathbb{R} \times \{0\}.$$

The first two characteristics equations are

$$\dot{x}(t) = x(t)\ x(0) = x_0 \in \mathbb{R}^2,$$
$$\dot{y}(t) = 2\ y(0) = y_0 \in \mathbb{R}^2.$$

The general solutions are $x(t) = x_0 e^t$ and $y(t) = 2t$.
Let $(x, y) \in \mathbb{R}^2$ be arbitrary. Consider the equation

$$x(t_0) = x_0 e^{t_0} = x, \quad y(t_0) = 2t_0 = y \tag{3}$$

The equation (3) has a unique solution. This is $t_0 = \frac{y}{2}$ and $(x_0, y_0) = (xe^{-\frac{y}{2}}, 0)$.

For this initial point $(x_0, y_0) = (xe^{-\frac{y}{2}}, 0)$ and $u_0(x_0, y_0) = \sin\left(xe^{-\frac{y}{2}}\right)$, we consider the third characteristics equation

$$\frac{d}{dt}z(t) = z(t), \quad z(0) = \sin\left(xe^{-\frac{y}{2}}\right).$$

The solution is

$$z(t) = \sin\left(xe^{-\frac{y}{2}}\right) e^t$$

and at $t_0 = \frac{y}{2}$ we get

$$z(t_0) = \sin\left(xe^{-\frac{y}{2}}\right) e^{\frac{y}{2}} =: u(x, y).$$

Temporal Equations

Consider a linear PDE of order 2 of the form

$$u_t + \sum_{i=1}^{n} a_i(\mathbf{x}) u_{x_i} = c_1(\mathbf{x})u + c_2(\mathbf{x}) \ \text{ in } \mathbb{R}^n. \tag{4}$$

In this case the auxiliary condition is (usually) given as an initial condition at time $t = 0$:

$$u_0(\mathbf{x}) = g(\mathbf{x}) \ \text{ on } \mathbb{R}^n.$$

The characteristics ODE for the t variable is always $\frac{d}{ds}t(s) = 1, \ t(0) = 0$. Thus $t = s$.
We have $\gamma_{(\mathbf{x}_0,0)}(t) = (t, x_1(t), \dots, x_n(t))$ and denote $(x_1(t), \dots, x_n(t)) =: \gamma_{\mathbf{x}_0}(t)$ for the characteristics.

If we set $V(\mathbf{x}) = (a_1(\mathbf{x}), \dots, a_n(\mathbf{x}))$, the PDE becomes

$$u_t + V(\mathbf{x}) \cdot \nabla u = c_1(\mathbf{x})u + c_2(\mathbf{x}).$$

The curves $\gamma_{\mathbf{x}_0}$ solve $\dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t)$ with $\gamma_{\mathbf{x}_0}(0) = \mathbf{x}_0$, hence are the flow curves of V.

If the flow map Φ_t of V is a diffeomorphism of $\mathbb{R}^n$ for every $t \geq 0$, then for $t >$ and for every $\mathbf{x} \in \mathbb{R}^n$ we can solve

$$\Phi_t(\mathbf{x}_0) = \mathbf{x} \iff \Phi_t^{-1}(x) = \mathbf{x}_0$$

uniquely. Hence, $\Phi_t(\mathbf{x}_0) = \gamma_{\mathbf{x}_0}(t) = \mathbf{x}$. In this case we can solve the characteristics equation for z

$$\frac{d}{dt} z_{\mathbf{x}_0}(t) = c_1(\gamma_{\mathbf{x}_0}(t))z(t) + c_2(\gamma_{\mathbf{x}_0})$$

with initial value $z_{\mathbf{x}_0}(0) = g(x_0)$ and define $u(\mathbf{x}) := z_{\mathbf{x}_0}(t) = z_{\Phi_t^{-1}(\mathbf{x})}(t)$ that is a solution for (4).
Note that u is indeed smooth enough.