MAT351 Partial Differential Equations
Lecture 4

September 23, 2020
Introduction to the Method of Characteristics

Last Lecture:
We found the general solution of $au_x + bu_y = 0$. Solutions are constant on lines parallel to $(a, b)$.

As another example we consider

$$u_x + yu_y = 0 \iff (1, y) \cdot \nabla u = 0 \quad \text{in } \mathbb{R}^2$$

with the auxiliary condition $u(0, y) = g(y)$ for $g \in C^1(\mathbb{R})$.

Instead of straight lines, now we are looking for curves $(x, y(x))$ such that

$$\frac{d}{dx} (x, y(x)) = (1, y) \iff \frac{d}{dx} x = 1 \& \frac{d}{dx} y = y$$

Hence $y(x) = y_0 e^x$ with $y(0) = y_0$.

Moreover, a solution $u$ of the PDE satisfies along the curve $(x, y(x))$:

$$\frac{d}{dx} u(x, y(x)) = \nabla u \cdot (1, y) = u_x + yu_y = 0$$

and $u(x, y(x)) = u(0, y(0)) = u(0, y_0)$ is independent of $x$.

Given a point $(\hat{x}, \hat{y}) \in \mathbb{R}^2$ we want to find $y_0$ and $y(\cdot)$ with $y(0) = y_0$ and $y(\hat{x}) = \hat{y}$.

Then, we know the value of $u$ in $(\hat{x}, \hat{y})$: It is

$$u(\hat{x}, \hat{y}) = u(\hat{x}, y(\hat{x})) = u(0, y_0) = g(y_0).$$

But by the formula for $y(x)$, we can indeed find such $y_0$: it is $y_0 = \hat{y} e^{-\hat{x}}$. Therefore we can write

$$u(\hat{x}, \hat{y}) = g(\hat{y} e^{-\hat{x}}).$$

This $u$ satisfies the PDE with the given auxiliary condition. (let us drop $\hat{}$) Indeed

$$u_x + yu_y = g'(ye^{-x})(-ye^{-x}) + yg'(ye^{-x})e^{-x} = 0.$$
Method of characteristics for linear equations

We consider a general linear PDE of order 1 with 2 independent variables $x, y$:

$$a(x, y)u_x + b(x, y)u_y = c_1(x, y)u + c_2(x, y) \text{ in } \mathbb{R}^2$$

(1)

where $a, b, c_1, c_2 \in C^2(\mathbb{R}^2)$. Assume there is an auxiliary condition given as follows.

$$u(x, y) = u_0(x, y) \text{ for } (x, y) \in \Gamma$$

where $\Gamma$ is a 1 dimensional subset in $\mathbb{R}^2$ and $u_0 : \Gamma \to \mathbb{R}$.

We can write (1) also as equation for directional derivatives:

$$V(x, y) \cdot \nabla u = c_1(x, y)u + c_2(x, y)$$

for a vectorfield $(x, y) \mapsto V(x, y) = (a(x, y), b(x, y)) \in \mathbb{R}^2$.

More generally, consider a general linear PDE of order 1 with $n$ independent variables:

$$\sum_{i=1}^{n} a_i(x) u_{x_i} = c_1(x)u + c_2(x) \text{ in } \mathbb{R}^n.$$  

(2)

where $a_i, c_1, c_2 \in C^2(\mathbb{R}^n)$ with an auxiliary condition

$$u(x) = u_0(x) \text{ for } x \in \Gamma$$

where $\Gamma$ is a $n - 1$ dimensional subset in $\mathbb{R}^n$.

Again we can write (2) as

$$V(x) \cdot \nabla u = c(x)u + d(x) \text{ for the vectorfield } V(x) = (a_1(x), \ldots, a_n(x)).$$
Method of characteristics for linear equations, continued

We want to find the flow curves of $V$. That is, we have to solve the following ODE:

$$\frac{d}{dt} \gamma_{x_0}(t) = \dot{\gamma}_{x_0}(t) = V(x), \quad \gamma_{x_0}(0) = x_0.$$ 

Recall that

$$\gamma_x(t) = \left( \begin{array}{c} \gamma_1^x(t) \\ \vdots \\ \gamma_n^x(t) \end{array} \right), \quad \dot{\gamma}_x(t) = \frac{d}{dt} \gamma_x(t) = \left( \begin{array}{c} \frac{d}{dt} \gamma_1^x(t) \\ \vdots \\ \frac{d}{dt} \gamma_n^x(t) \end{array} \right).$$

Assume we have a solution $u \in C^1(\mathbb{R}^n)$. How does $u$ evolve along a flow curve $\gamma_{x_0}$?

The chain rule yields

$$\frac{d}{dt} u \circ \gamma_{x_0}(t) = \nabla u(\gamma_{x_0}(t)) \cdot \dot{\gamma}_{x_0}(t) = \nabla u(\gamma_{x_0}(t)) \cdot V(\gamma_{x_0}(t)).$$

On the other hand, by the PDE we have

$$\nabla u(\gamma_{x_0}(t)) \cdot V(\gamma_{x_0}(t)) = c_1(\gamma_{x_0}(t)) u(\gamma_{x_0}(t)) + c_2(\gamma_{x_0}(t)).$$

This gives us an ODE for the composition $u \circ \gamma_{x_0}(t) =: z_{x_0}(t)$:

$$\frac{d}{dt} z_{x_0}(t) = \dot{z}_{x_0}(t) = c_1(\gamma_{x_0}(t)) z_{x_0}(t) + c_2(\gamma_{x_0}(t)), \quad z_{x_0}(0) = u_0(\gamma_{x_0}(0)).$$
Characteristics equations

Definition

Consider linear PDE of order 1 with \( n \) independent variables in the form

\[
V(x) \nabla u = c_1(x)u + c_2(x) \quad \text{in } \mathbb{R}^n \quad \text{and } u = u_0 \quad \text{on } \Gamma
\]

where \( \Gamma \) is an \( n-1 \) dimensional subset.

The corresponding characteristics equations are

\[
\begin{align*}
\dot{\gamma}_{x_0}(t) &= V \circ \gamma_{x_0}(t) \\
\dot{z}_{x_0}(t) &= c_1(\gamma_{x_0}(t))z_{x_0}(t) + c_2(\gamma_{x_0}(t)) \quad z_{x_0}(0) = u_0(x_0).
\end{align*}
\]

For the case of 2 independent variable the systems of equations becomes

\[
\begin{align*}
\dot{x}(t) &= a(x(t), y(t)), \quad x(0) = x_0, \\
\dot{y}(t) &= b(x(t), y(t)), \quad y(0) = y_0, \\
\dot{z}(t) &= c_1(x(t), y(t))z(t) + c_2(x(t), y(t)), \quad z(0) = u_0(x_0, y_0).
\end{align*}
\]

Question

How we obtain a solution for the PDE?
Solving the PDE

We will determine the value $u(x)$ for every $x \in \mathbb{R}^n$ using the method of characteristics.

From the previous consideration, we know that the solution for the PDE (2) must obey the characteristics equations.

Our strategy is:
For $x \in \mathbb{R}^n$ arbitrary we pick a characteristic $\gamma_{x_0}$ for an initial point $x_0 \in \Gamma$ such that $\gamma_{x_0}(t_0) = x$ for some $t_0 \in [0, \infty)$.
Then we can solve the characteristic equation for $z_{x_0} = u \circ \gamma_{x_0}$

$$\dot{z}_{x_0}(t) = c_1 \circ \gamma_{x_0}(t) z_{x_0}(t) + c_2 \circ \gamma_{x_0}(t) \quad \text{with} \quad z_{x_0}(0) = u_0(x_0)$$

and set

$$z_{x_0}(t_0) = u \circ \gamma_{x_0}(t_0) = u(x).$$

Hence, if for every $x \in \mathbb{R}^n$ we can find a unique $x_0 \in \Gamma$ and $t_0 \geq 0$ such that

$$\gamma_{x_0}(t_0) =: \Phi(x_0, t_0) = x$$

we can define a function $u : \mathbb{R}^n \to \mathbb{R}$ that will solve the PDE and satisfies the auxiliary condition (by construction). Indeed we have

**Proposition**

*Assuming the function $u : \mathbb{R}^n \to \mathbb{R}$ defined above is a $C^1$ function, then it solves*

$$\nabla u \cdot V(x) = c_1(x) u + c_2(x) \quad \text{in} \quad \mathbb{R}^n \quad \text{with} \quad u(x) = u_0(x) \quad \text{for} \quad x \in \Gamma.$$
Proof: Let \( x \in \mathbb{R}^n \), assume \( x_0 \) is the unique point such that: \( \gamma_{x_0} \) solves \( \dot{\gamma}_{x_0}(t) = V \circ \gamma_{x_0}(t) \) with \( \gamma_{x_0}(t_0) = x \).
Then \( u(x) \) is defined as \( z_{x_0}(t_0) \).
First, if \( x \in \Gamma \), we pick \( x_0 = x \), \( \gamma_{x_0} = \gamma_x \) and \( t_0 = 0 \). Then \( u(x) = z_x(0) = u_0(x) \).
For general \( x \in \mathbb{R}^n \) we compute
\[
\nabla u(x) \cdot V(x) = \nabla u(\gamma_{x_0}(t_0)) \cdot \dot{\gamma}_{x_0}(t_0) = \frac{d}{dt} u \circ \gamma_{x_0}(t_0).
\]
Since \( z_{x_0} \) solves the last characteristic equation, the right hand side is equal to
\[
c_1(\gamma_{x_0}(t_0)) u \circ \gamma_{x_0}(t_0) + c_2(\gamma_{x_0}(t_0)) = c_1(x) u(x) + c_2(x).
\]
Hence \( u \) indeed solves the equation. \( \square \)

Remark
Let us summarize what we assumed here
- We need that any flow curve meets \( \Gamma \) in exactly one point.
  For any \( x \) there exists a unique flow curve \( \gamma_{x_0} \) such that \( x_0 \in \Gamma \) and \( \gamma_{x_0}(t_0) = x \).
  Then, we can solve the initial value problem for \( z_{x_0} \) because the initial value is given by \( u(x_0) = u_0(x_0) \).
  In other words, we have to solve the equation
  \[
  \Phi_{t_0}|_{\Gamma}(x_0) = x
  \]
  where \( \Phi_t(y) = \gamma_y(t) \) is the flow map of \( V \), and \( \Phi_t|_{\Gamma} \) is the restriction of \( \Phi_t \) to \( \Gamma \).
- We need that \( u \in C^1(\mathbb{R}^n) \).
Example

We find the solution for

\[ xu_x + 2u_y = 3u \text{ in } \mathbb{R}^2, \quad u(x, 0) = \sin x, \quad \Gamma = \mathbb{R} \times \{0\}. \]

The first two characteristics equations are

\[ \dot{x}(t) = x(t), \quad x(0) = x_0 \in \mathbb{R}^2, \]
\[ \dot{y}(t) = 2, \quad y(0) = y_0 \in \mathbb{R}^2. \]

The general solutions are \( x(t) = x_0 e^t \) and \( y(t) = 2t \).

Let \((x, y) \in \mathbb{R}^2\) be arbitrary. Consider the equation

\[ x(t_0) = x_0 e^{t_0} = x, \quad y(t_0) = 2t_0 = y \tag{3} \]

The equation (3) has a unique solution. This is \( t_0 = \frac{y}{2} \) and \((x_0, y_0) = (xe^{-\frac{y}{2}}, 0)\).

For this initial point \((x_0, y_0) = (xe^{-\frac{y}{2}}, 0)\) and \( u_0(x_0, y_0) = \sin \left(xe^{-\frac{y}{2}}\right) \), we consider the third characteristics equation

\[ \frac{dz(t)}{dt} = z(t), \quad z(0) = \sin \left(xe^{-\frac{y}{2}}\right). \]

The solution is

\[ z(t) = \sin \left(xe^{-\frac{y}{2}}\right) e^t \]

and at \( t_0 = \frac{y}{2} \) we get

\[ z(t_0) = \sin \left(xe^{-\frac{y}{2}}\right) e^{\frac{y}{2}} =: u(x, y). \]
Temporal Equations
Consider a linear PDE of order 2 of the form
\[ u_t + \sum_{i=1}^{n} a_i(x)u_{x_i} = c_1(x)u + c_2(x) \quad \text{in } \mathbb{R}^n. \] (4)

In this case the auxiliary condition is (usually) given as an initial condition at time \( t = 0 \):
\[ u_0(x) = g(x) \quad \text{on } \mathbb{R}^n. \]

The characteristics ODE for the \( t \) variable is always \( \frac{dt}{ds} = 1, \quad t(0) = 0 \). Thus \( t = s \).
We have \( \gamma_{(x_0,0)}(t) = (t, x_1(t), \ldots, x_n(t)) \) and denote \( (x_1(t), \ldots, x_n(t)) =: \gamma_{x_0}(t) \) for the characteristics.
If we set \( V(x) = (a_1(x), \ldots, a_n(x)) \), the PDE becomes
\[ u_t + V(x) \cdot \nabla u = c_1(x)u + c_2(x). \]

The curves \( \gamma_{x_0} \) solve \( \dot{\gamma}_{x_0}(t) = V \circ \gamma_{x_0}(t) \) with \( \gamma_{x_0}(0) = x_0 \), hence are the flow curves of \( V \).
If the flow map \( \Phi_t \) of \( V \) is a diffeomorphism of \( \mathbb{R}^n \) for every \( t \geq 0 \), then for \( t > 0 \) and for every \( x \in \mathbb{R}^n \) we can solve
\[ \Phi_t(x_0) = x \iff \Phi_t^{-1}(x) = x_0 \]
uniquely. Hence, \( \Phi_t(x_0) = \gamma_{x_0}(t) = x \). In this case we can solve the characteristics equation for \( z \)
\[ \frac{dz_{x_0}(t)}{dt} = c_1(\gamma_{x_0}(t))z(t) + c_2(\gamma_{x_0}) \]
with initial value \( z_{x_0}(0) = g(x_0) \) and define \( u(x) := z_{x_0}(t) = z_{\Phi_t^{-1}(x)}(t) \) that is a solution for (4).
Note that \( u \) is indeed smooth enough.