MAT351 Partial Differential Equations
Lecture 5

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Method of Characteristics, Summary

The method of characteristics is a recipe to solve a linear PDE of order 1 in several variables:

$$\nabla u \cdot V(\mathbf{x}) = c_1(\mathbf{x})u + c_2(\mathbf{x}) \text{ in } \Omega \subset \mathbb{R}^n \;\; \& \;\; \text{auxiliary condition: } u(\mathbf{x}) = u_0(\mathbf{x}) \text{ on } \Gamma.$$

$V(\mathbf{x}) = (a_1(\mathbf{x}), \dots, a(\mathbf{x})) \in C^1(\Omega, \mathbb{R}^n)$, $c_1, c_2 \in C^1(\Omega)$, and Γ is a $n-1$ dimensional subset in Ω.

The **recipe** goes as follows

- Assuming the existence of a C^1 solution, we deduced the *characteristics equations*:

$$\begin{cases} \dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t) & \gamma_{\mathbf{x}_0}(0) = \mathbf{x}_0 \in \Gamma, \\ \dot{z}_{\mathbf{x}_0}(t) = c_1(\gamma_{\mathbf{x}_0}(t)) z_{\mathbf{x}_0}(t) + c_2(\gamma_{\mathbf{x}_0}(t)) & z_{\mathbf{x}_0}(0) = u_0(\mathbf{x}_0). \end{cases}$$

  This system of ODEs can be solved uniquely on a maximal intervall $(\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0}) \ni 0$ (General Existence and Uniqueness Theorem for ODEs).

- We note that the PDE is a statement about the directional derivatives of a C^1 solution u. Precisely, given a flow curve $\gamma_{\mathbf{x}_0}$ of V with $\mathbf{x}_0 \in \Gamma$ and the value of u in $\mathbf{x}_0$, the equation for $z_{\mathbf{x}_0}$ gives the directional derivatives of u along $\gamma_{\mathbf{x}_0}$.
- The number $z_{\mathbf{x}_0}(t)$ tells us the values of u at $\gamma_{\mathbf{x}_0}(t)$ for $t \in (\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0})$.

The only result that this analysis actually proves is:

Proposition

If *$u \in C^1(\Omega)$ solves the PDE and $\gamma_{\mathbf{x}_0} : (\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0}) \to \mathbb{R}^n$ is a flow curve of V with $\mathbf{x}_0 \in \Gamma$,*
then *$u \circ \gamma_{\mathbf{x}_0} : (\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0}) \to \mathbb{R}$ with $u \circ \gamma_{\mathbf{x}_0}(0) = u_0(\mathbf{x}_0)$ must solve the ODE for $z_{\mathbf{x}_0}$.*

This gives us a method to "synthesize" an explicite solution via the following steps:

If we can find a unique solution $(\mathbf{x}_0, t_0)$ for the equation $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$ for every $\mathbf{x} \in \Omega$.
And $(\mathbf{x}_0, t_0)$ depends in a sufficiently smooth way on $\mathbf{x}$.

Remark

However, this might not be always possible:

There might exist $\mathbf{x} \in \Omega$ for which there exists not solution of $\gamma_{\mathbf{x}_0}(t_0) = x$ with $\mathbf{x}_0 \in \Gamma$ and $t \in (\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0})$.

Temporal Equations, revisited

Given a linear PDE of order 2 of the form

$$u_t + \sum_{i=1}^{n} a_i(\mathbf{x}) u_{x_i} = c_1(\mathbf{x})u + c_2(\mathbf{x}) \ \text{ in } \mathbb{R}^n \times \mathbb{R} \tag{1}$$

The initial condition at time $t = 0$ is

$$u_0(\mathbf{x}) = g(\mathbf{x}) \ \text{ on } \mathbb{R}^n, \ g \in C^1(\mathbb{R}^n).$$

The characteristics ODE for the t variable is always $\frac{d}{ds}t(s) = 1, \ t(0) = 0$. Thus $t = s$.
If we set $V(\mathbf{x}) = (a_1(\mathbf{x}), \ldots, a_n(\mathbf{x}))$, the PDE becomes

$$u_t + V(\mathbf{x}) \cdot \nabla u = c_1(\mathbf{x})u + c_2(\mathbf{x}).$$

The flow curves $\gamma_{\mathbf{x}_0}$ of V are $\dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t)$ with $\gamma_{\mathbf{x}_0}(0) = \mathbf{x}_0$.
The characteristics of the PDE are

$$\gamma_{(\mathbf{x}_0,0)}(t) = \begin{pmatrix} \gamma_{\mathbf{x}_0}(t) \\ t \end{pmatrix}.$$

Applying our recipe means to solve $\gamma_{\mathbf{x}_0}(t) = \mathbf{x}$ uniquely for every $\mathbf{x} \in \mathbb{R}^n$ and every $t \in \mathbb{R}$.

If the flow map $\Phi_t(\mathbf{x}) = \gamma_{\mathbf{x}}(t)$ of V is a diffeomorphism of $\mathbb{R}^n$ for every $t \geq 0$, then

$$\Phi_t^{-1}(x) = \mathbf{x}_0 \ \text{ solves } \ \Phi_t(\mathbf{x}_0) = \mathbf{x}$$

uniquely. In this case we can define $u(\mathbf{x}) := z_{\mathbf{x}_0}(t) = z_{\Phi_t^{-1}(\mathbf{x})}(t)$ that is a solution for (1).

Example: Transport equation with constant coefficients

Consider

$$u_t + \sum_{i=1}^{n} a_i u_{x_i} = 0 \text{ in } \mathbb{R}^n \times [0, \infty) \quad \text{with} \quad u(\mathbf{x}, 0) = g(\mathbf{x}) \text{ on } \mathbb{R}^n, \; g, f \in C^1(\mathbb{R}^n).$$

Define $V(\mathbf{x}) \equiv (a_1, \ldots, a_n) = v$. The flow curves of V are

$$\gamma_{\mathbf{x}_0} = \mathbf{x}_0 + tv$$

and the flow map $\phi_t(\mathbf{x}_0) = \mathbf{x}_0 + tv$ is a diffeomorphism. Hence

$$\mathbf{x}_0 := \phi_t^{-1}(\mathbf{x}) = \mathbf{x} - tv \;\text{ uniquely solves }\; \phi_t(\mathbf{x}_0) = x.$$

Solving the characteristics equation

$$\frac{d}{dt} z_{\mathbf{x}_0}(t) = 0, \quad z_{\mathbf{x}_0}(0) = u(\mathbf{x}_0, 0) = g(\mathbf{x}_0)$$

yields $z_{\mathbf{x}_0}(t) = g(\mathbf{x}_0)$. Hence $u(\mathbf{x}, t) = g(\mathbf{x} - tv)$ is the solution for the PDE.

Semi-linear PDEs

Consider a semi-linear PDE of order 1

$$V(\mathbf{x}) \cdot \nabla u = c(\mathbf{x}, u) \;\text{ in }\; \Omega \subset \mathbb{R}^n \;\text{ with auxiliary condition }\; u(\mathbf{x}) = g(\mathbf{x}) \text{ on } \Gamma \subset \Omega.$$

The methods of characteristics applies in the exact same way.

But the equation for $z_{\mathbf{x}_0}$ becomes a nonlinear equation in $z_{\mathbf{x}_0}$:

$$\frac{d}{dt} z_{\mathbf{x}_0}(t) = c(\gamma_{\mathbf{x}_0}(t), z_{\mathbf{x}_0}(t)), \quad z_{\mathbf{x}_0}(0) = g(\mathbf{x}_0), \; \mathbf{x}_0 \in \Gamma.$$

Example (Transport equation with nonlinear right hand side)

Consider

$$u_x + u_y = u^2 \text{ on } \Omega \subset \mathbb{R}^2 \;\text{ with } u(\cdot, 0) = g \in C^1(\mathbb{R}).$$

Here the vector field is $V(x) = (1, 1)$ with the flow $\gamma_{(x_0,0)}(t) = (x_0 + t, y_0 + t)$.

Hence for $(x, y) \in \mathbb{R}^2$ the point $(x_0, 0) = (x - t_0, 0)$ and $t_0 = y$ solves $\gamma_{(x_0,0)}(t_0) = (x, y)$.
The characteristics equation for z_{x_0} is

$$\frac{d}{dt} z_{(x_0,y_0)} = (z_{(x_0,y_0)})^2 \;\; z_{(x_0,y_0)}(0) = g(x_0).$$

The solution of this ODE Is $z_{(x_0,y_0)}(t) = \frac{1}{\frac{1}{g(x_0)} - t}$. So $u(x, y) = u_{(x_0,0)}(t_0) = \frac{1}{\frac{1}{g(x-y)} - y}$.

This yields the following contraint: $g(x - y)y < 1$. Hence, to find a solution it is necessary that $\Omega \subset \{(x, y) \in \mathbb{R}^2 : g(x - y)y \leq 1\}$.

Quasi-linear PDEs

Consider a quasilinear PDE of order 1

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i} = c(\mathbf{x}, u) \text{ in } \Omega \subset \mathbb{R}^n \text{ with auxiliary condition } u(\mathbf{x}) = g(\mathbf{x}) \text{ on } \Gamma \subset \Omega.$$

Defining the vector field $V(\mathbf{x}, u) = (a_1(\mathbf{x}, u), \ldots, a_n(\mathbf{x}, u))$ the PDE becomes

$$V(\mathbf{x}, u) \cdot \nabla u = c(\mathbf{x}, u).$$

Assuming a sufficiently smooth solution u we can write down the following equations

$$\dot{\gamma}_{\mathbf{x}_0}(t) = V(\gamma_{\mathbf{x}_0}(t), u \circ \gamma_{\mathbf{x}_0}(t))$$
$$\frac{d}{dt} u \circ \gamma_{\mathbf{x}_0}(t) = c(\gamma_{\mathbf{x}_0}(t), u \circ \gamma_{\mathbf{x}_0}(t))$$

Not that in contrast to linear and semi-linear PDEs this is a coupled system of ODEs.

Provided the coefficients are a_i and c are C^1 the solution $(\gamma_{\mathbf{x}_0}(t), z_{\mathbf{x}_0}(t))$ exists and depends in C^1 sense on $(\mathbf{x}_0, t)$.

Transversality condition, Existence of local solutions

Consider again

$$au_x + bu_y = 0 \ \text{ on } \mathbb{R}^2 \quad \text{with } u(0,y) = g(y),\ g \in C^1(\mathbb{R}).$$

We could construct a (unique) solution as long as $a \neq 0$. Or

$$\det \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \neq 0.$$

Consider a general quasi-linear PDE of order 1 in **two independent variables**

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \text{ in } \Omega \subset \mathbb{R}^2 \text{ \& auxiliary condition } u(x,y) = g(x,y) \text{ on } \Gamma$$

where $\Gamma = \text{Im}\eta$ and $\eta : \mathbb{R} \to \mathbb{R}^2, \eta(t) = (k(t), l(t))$, $\eta \in C^1(J, \mathbb{R}^2)$ for an interval $J \subset \mathbb{R}$.

The Transversality Condition for this problem is

$$\det \begin{pmatrix} a(k(t), l(t), g(k(t), l(t))) & \dot{k}(t) \\ b(k(t), l(t), g(k(t), l(t)) & \dot{l}(t) \end{pmatrix} \neq 0.$$

Theorem

Consider the previous quasi-linear PDE and assume the transversality condition. Then, for every $s_0 \in J$ there exists $\delta > 0$ such that $B_\delta(\eta(s_0)) \subset \Omega$ and

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \text{ in } B_\delta(\gamma(t)) \ \text{ with } u(x,y) = g(x,y) \text{ on } \Gamma \cap B_\delta(t)$$

has a unique solution u.

Proof

We already mentioned that $\gamma_{x_0,y_0}(t) =: \phi(x_0, y_0, t)$ dependes smoothly on (x_0, y_0, t) for $(x_0, y_0, t) \in \mathbb{R}^2 \times \mathbb{R}$.

We pick $s_0 \in J$ with $\eta(s_0) = x_0$ and define

$$\psi(s,t) = \phi(\eta(s), t), \;\; s \in (s_0 - \delta, s_0 + \delta) \subset J, t \in (-\epsilon, \epsilon),$$

where we choose $\delta > 0$ such that $(-\epsilon, \epsilon) \subset (\alpha_{\eta(s)}, \omega_{\eta(s)})$ for all $s \in (s_0 - \delta, s_0 + \delta)$.

Then $\psi : (s_0 - \delta, s_0 + \delta) \times (-\epsilon, \epsilon) \to \mathbb{R}$ is C^1, since it is a composition of C^1 maps.

We compute

$$\frac{\partial}{\partial t}\psi(s,t)\Big|_{s_0,0} = \frac{d}{dt}\Big|_{t=0}\phi(\eta(s_0), t) = (a(k(s_0), l(s_0), z_{k(s_0),l(s_0)}), b(k(s_0), l(s_0), z_{k(s_0),l(s_0)}))$$

and

$$\frac{d}{ds}\psi(s,t)\Big|_{s_0,0} = \frac{d}{ds}\eta(s_0, 0) = \frac{d}{ds}\phi(\eta(s_0), 0) = \frac{d}{ds}\eta(s_0) = (\dot{k}(s_0), \dot{l}(s_0)).$$

Now, the transversality condition implies that the differential of the map $(s,t) \mapsto \phi(\eta(s), t)$ in $(s_0, 0)$ is invertible.

Hence, by the inverse function theorem, there exists a smaller $\delta > 0$ such that $\psi(s,t)$ is a C^1-diffeomorphism on $(s_0 - \delta, s_0 + \delta) \times (-\delta, \delta)$.

Hence $\Phi((-\delta, \delta)^2) =: U \subset \mathbb{R}^2$ is an open domain in $\mathbb{R}^2$ and for all $(x,y) \in U$ there exists a unique pair (s,t) such that $\phi(\eta(s), t) = (x,y)$. □