MAT351 Partial Differential Equations
Lecture 6

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1.3 Transversality condition, Existence of local solutions

Consider a general quasi-linear PDE of order 1 in **two independent variables**

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \text{ in } \Omega \subset \mathbb{R}^2 \text{ \& auxiliary condition } u(x,y) = g(x,y) \text{ on } \Gamma$$

where $\Gamma = \mathrm{Im}\eta$ and $\eta : \mathbb{R} \to \mathbb{R}^2, \eta(t) = (k(t), l(t))$, $\eta \in C^1(J, \mathbb{R}^2)$ for an interval $J \subset \mathbb{R}$. Assume Γ is a embedded sumanifold.

The Transversality Condition for this problem is

$$\det \begin{pmatrix} a(k(t), l(t), g(k(t), l(t))) & \dot{k}(t) \\ b(k(t), l(t), g(k(t), l(t)) & \dot{l}(t) \end{pmatrix} \neq 0.$$

Theorem

Consider the previous quasi-linear PDE and assume the transversality condition. Then, for every $s_0 \in J$ there exists $\delta > 0$ such that $B_\delta(\eta(s_0)) \subset \Omega$ and

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \text{ in } B_\delta(\gamma(t)) \quad \text{with } u(x,y) = g(x,y) \text{ on } \Gamma \cap B_\delta(t)$$

has a unique solution u.

Proof of the Theorem

Recall the characteristics equations

$$\frac{d}{dt}\gamma_{x_0,y_0}(t) = \begin{pmatrix} a(\gamma_{x_0,y_0}(t), z_{x_0,y_0}(t)) \\ b(\gamma_{x_0,y_0}(t), z_{x_0,y_0}(t)) \end{pmatrix}, \quad \gamma_{x_0,y_0}(0) = (x_0, y_0),$$
$$\frac{d}{dt} z_{x_0,y_0}(t) = c(\gamma_{x_0,y_0}(t), z_{x_0,y_0}(t)), \quad z_{x_0,y_0}(0) = g(x_0, y_0).$$

We already mentioned that $\gamma_{x_0,y_0}(t) =: \phi(x_0, y_0, t)$ dependes smoothly on (x_0, y_0, t).

We pick $s_0 \in J$ with $\eta(s_0) = (x_0, y_0)$ and define

$$\psi(s,t) = \phi(\eta(s), t), \ s \in (s_0 - \delta, s_0 + \delta) \subset J, t \in (-\epsilon, \epsilon),$$

where we choose $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset (\alpha_{\eta(s)}, \omega_{\eta(s)})$ for all $s \in (s_0 - \delta, s_0 + \delta)$.

Then $\psi : (s_0 - \delta, s_0 + \delta) \times (-\epsilon, \epsilon) \to \mathbb{R}^2$ is C^1, since it is a composition of C^1 maps.

We compute

$$\frac{\partial}{\partial t}\psi(s,t)\Big|_{s_0,0} = \frac{d}{dt}\Big|_{t=0}\phi(\eta(s_0), t) = (a(k(s_0), l(s_0), z_{k(s_0),l(s_0)}), b(k(s_0), l(s_0), z_{k(s_0),l(s_0)}))$$

$$\frac{\partial}{\partial s}\psi(s,t)\Big|_{s_0,0} = \frac{d}{ds}\Big|_{s=s_0}\phi(\eta(s), 0) = \frac{d}{ds}\Big|_{s_0}\eta(s) = (\dot{k}(s_0), \dot{l}(s_0)).$$

By the transversality condition the differential of $(s,t) \mapsto \phi(\eta(s), t)$ in $(s_0, 0)$ is invertible.

Hence, by the inverse function theorem, there exists a smaller $\hat{\delta} > 0$ such that $\psi(s,t)$ is a C^1-diffeomorphism on $(s_0 - \hat{\delta}, s_0 + \hat{\delta}) \times (-\hat{\delta}, \hat{\delta})$.

Hence $\psi((s_0 - \hat{\delta}, s_0 + \hat{\delta}) \times (-\hat{\delta}, \hat{\delta})) =: U \subset \mathbb{R}^2$ is an open domain in $\mathbb{R}^2$ and for all $(x,y) \in U$ there exists a unique pair (s,t) such that $\phi(\eta(s), t) = (x,y)$. $\square$

1.4 Burgers' Equation

We studied the linear transport equation: $u_t + cu_x = 0$ in $\mathbb{R} \times [0, \infty)$, $u(x,0) = g(x)$.

The solution was given by $u(x,t) = g(x - tc)$.

This is not a good model for describing natural phenomenas like waves, or street traffic.

A better equation is

The inviscid Burgers' equation

$$u_t + uu_x = 0 \text{ in } \mathbb{R} \times [0, \infty), \; u(x,0) = g(x), \; g \in C^1(\mathbb{R}).$$

Because of the non-linearity uu_x the equation is a **quasi-linear equation**.

A solution u moves with speed in x that is given by the value of u in x itself.

We can easily write down the characteristics equations:

$$\begin{aligned} \frac{d}{dt}\gamma_{x_0}(t) &= z_{x_0}, \quad \gamma_{x_0}(0) = x_0 \in \mathbb{R}, \\ \frac{d}{dt}z_{x_0}(t) &= 0, \qquad z_{x_0}(0) = g(x_0). \end{aligned}$$

The solution of this system is

$$\begin{aligned} z_{x_0}(t) &\equiv g(x_0) \qquad\quad \text{for } t \geq 0, \\ \gamma_{x_0}(t) &= tg(x_0) + x_0 \text{ for } t \geq 0. \end{aligned}$$

Hence the characteristic that starts in $x_0 (= (x_0, 0))$ is the straight line with slope $g(x_0)$.

If g is an increasing function, then the corresponding characteristics will span out the space-time half plane $\mathbb{R} \times [0, \infty)$.

However, if g is not increasing, then characteristics will collide.

Example

$$u_t + uu_x = 0 \text{ in } \mathbb{R} \times [0, \infty) \;\; \& \;\; u(x, 0) = g(x), \; g \in C^1(\mathbb{R})$$

with

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ \text{decreasing} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

For simplicity, we set $g(x) = 1 - x$ in $x \in (0, 1)$ (although g is then not C^1).

We can try to solve this equation with the characteristics method:

We observe

- Around $(\hat{x}, 0)$ with $\hat{x} \leq 0$, we have $\gamma_{x_0}(t) = t + x_0$ and the solution $u(x, t) \equiv 1$,
- Around $(\hat{x}, 0)$ with $\hat{x} \geq 1$, we have $\gamma_{x_0}(t) = x_0$ and the solution is $u(x, t) \equiv 0$,
- For $\hat{x} \in (0, 1)$ the following happens:

  If (x, t) satisfies $t < x < 1$, we solve $\gamma_{x_0}(t) = t(1 - x_0) + x_0 = x$: $\;x_0 = \frac{x-t}{1-t} \in (0, 1)$.

  Then, the slope of $\gamma_{x_0}(t)$ is $g(x_0) = (1 - x_0) = \frac{1-x}{1-t}$. That is also the value of $u(x, t)$.

1.5 Distributional solutions of scalar conservations laws

Let us consider a general

Scalar Conversation Law

$$u_t + (f(u))_x = u_t + f'(u)u_x = 0 \text{ in } \mathbb{R} \times [0, \infty) \ \& \ u(x,0) = g(x) \tag{1}$$

where $g \in C^1(\mathbb{R})$ and $f \in C^2(\mathbb{R})$ with $f'' \geq 0$ (hence f is convex).

From the Burgers Equation we learned that C^1 solutions may be defined only up to some time t^*. Then a **Shock** has developed.

Shocks can have a physical meaning, so it is desirable to extend our concept of solution to include functions u that are **discontinuous and still satisfy the PDE in a generalized (or weak) sense.**

Definition

We say $u(x,t), (x,t) \in \mathbb{R} \times [0, \infty)$ is piecewise smooth if

- u is C^1 in all points (x,t) except along a C^1 curve $s(t), t \in (a, \infty)$,
- u is discontinuous in $s(t)$ for all $t \in (a, \infty)$.

In addition we assume that for every $t \in (a, \infty)$ the limits

$$u^+(s(t),t) := \lim_{x \downarrow s(t)} u(x,t), \quad u^-(s(t),t) := \lim_{x \uparrow s(t)} u(x,t) \ \text{ exist.}$$

Question

What is a good notion of solution for the conservation law (1) in the class of piecewise smooth functions u?

Should we call a piecewise smooth function u already a solution if u solves the conversation law in the classical sense in every (x, t) where u is a C^1 function?

- Then answer to the second question is NO!
- When we derived the conversations law, we assumed a priori that a solution would be C^1.
- But eventually the class of C^1 function is to small to capture all physical meaningful events.

Recall we had the integral equation

$$\int_{-\infty}^{\infty} 1_{\Omega} \left[u_t + (f(u))_x\right] dx = 0$$

for every connected domain with smooth boundary $\Omega \subset \mathbb{R}$.

This motivates the following definition.

Definition (Distributional solutions)

We say a piecewise smooth function u is a solution of (1) in the distributional sense if

$$\int_0^{\infty} \int_{-\infty}^{\infty} \left[u\phi_t + f(u)\phi_x\right] dxdt = 0 \text{ for any } \phi \in C_c^{\infty}(\mathbb{R} \times (0, \infty)).$$

Using integration by parts and the fundamental theorem of calculus we see that for every $(x, t) \neq (s(t), t)$ the function u satisfies $u_t + (f(u))_x = 0$ classically.

Theorem (Rankine-Hugoniot jump condition(s))

Let $s(t), t \geq 0$ is a C^1 curve in $\mathbb{R} \times [0, \infty)$ (parametrized as graph). Assume u is piecewise smooth in the sense of the previous definition. Then u is a solution of (1) in the sense of distributions if and only if u is a classical solution in any point where u is C^1 and

$$s'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} \circ s(t) \quad \textit{for every } t \in (0, \infty).$$

Proof of the Theorem.

Consider u that is piecwise smooth in the previous sense.

Let ϕ be in $C_c^1(\mathbb{R} \times (0, \infty))$ with compact support in $B_r((s(t), t))$.

We define

$$B^+ = \{(x, t) \in B_r((s(t_0), t_0)) : x \geq s(t)\} \;\; \& \;\; B^- = \{(x, t) \in B_r((s(t_0), t_0)) : x \leq s(t)\}$$

Assume u is a solution in distributional sense. Then

$$0 = \int_0^\infty \int_{-\infty}^\infty [u\phi_t + f(u)\phi_x]\, dxdt = \int\int_{B^+} [u\phi_t + f(u)\phi_x]\, dxdt + \int\int_{B^-} [u\phi_t + f(u)\phi_x]\, dxdt$$

Now, here comes a trick. The identity

$$u\phi_t + f(u)\phi_x = u\phi_t + f(u)\phi_x + u_t\phi + (f(u))_x\phi$$

holds on $\{(x, t) : x > s(t)\}$ and $\{(x, t) : x < s(t)\}$.The right hand side in the previous identiy becomes

$$u\phi_t + f(u)\phi_x + u_t\phi + (f(u))_x\phi = (u\phi)_t + (f(u)\phi)_x = \nabla \cdot (f(u)\phi, u\phi)$$

Inserting this back into the integral identy yields

$$\int\int_{B^+}[u\phi_t+f(u)\phi_x]\,dxdt=\int\int_{B^+}\nabla\cdot(f(u)\phi,u\phi)dxdt$$

and the same for the integral over B^-.By the divergenc theorem (for domains with corners)

$$\int\int_{B^+}\nabla\cdot(f(u)\phi,u\phi)dxdt=\int_{\partial B^+}N\cdot(f(u^+)\phi,u^+\phi)dS=\int_{\{(s(t),t):t>0\}}\phi\left(N\cdot(f(u^+),u^+)\right)dS$$

$$\int\int_{B^+}\nabla\cdot(f(u^+)\phi,u^+\phi)dxdt=\int_0^\infty\phi(t)(-s'(t),1)\cdot(f(u^+),u^+)\circ(s(t),t)\sqrt{1+|s'(t)|^2}dt.$$

For the integral that involves B^- this is

$$\int\int_{B^-}\nabla\cdot(f(u^+)\phi,u^+\phi)dxdt=\int_0^\infty\phi(t)(s'(t),-1)\cdot(f(u^+),u^+)\circ(s(t),t)\sqrt{1+|s'(t)|^2}dt.$$

It follows that

$$\begin{aligned}0=\int_0^\infty\phi(t)\Big[&(-s'(t),1)\cdot(f(u^+),u^+)\circ(s(t),t)\\&+(s'(t),-1)\cdot(f(u^-),u^-)\circ(s(t),t)\Big]\sqrt{1+|s'(t)|^2}dt.\end{aligned}$$

We conclude that

$$0=-s'(t)u^++f(u^+)s'(t)u^--f(u^-)\ \rightarrow\ s'(t)=\frac{f(u^+)-f(u^-)}{u^+-u^-}\circ(s(t),t).$$

This is the jump condition. Now assuming u is a solution when it is C^1 together with jump condition, we can reverse this chain of implications and obtain u is solution in distributional sense. □

1.6 Non-uniquness of distributional solutions, Lax entropy condition

Since we know that for certain initial conditions, shocks always develop, and since we have the concept of distributional solution at hand, we consider the following PDE problem for the Burgers equation.

$$u_t + uu_x = 0 \text{ in } \mathbb{R} \times [0, \infty) \;\; \& \;\; u(x, 0) = g(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Let us apply the previous theorem. We want find a C^1 curve $s(t),\ t \geq 0$ that satisfies the jump condition. For the burgers equation we have $f(x) = \frac{1}{2}x^2$. Then the jump condition is

$$\frac{(u^+)^2 - (u^-)^2}{2(u^+ - u^-)} = \frac{1}{2}(u^+ - u^-) = s'(t).$$

Therefore, distributional solutions of the previous PDE with initial condition are

$$u(x, t) = \begin{cases} 1 & \text{for } x \leq \frac{1}{2}t, \\ 0 & \text{for } x > \frac{1}{2}t. \end{cases}$$

and

$$v(x, t) = \begin{cases} 0 & \text{for } x \leq \frac{1}{2}t, \\ 1 & \text{for } x > \frac{1}{2}t. \end{cases}$$

We want to choose the solution that is physical meaninful. Which one is it?