MAT351 Partial Differential Equations
Lecture 7

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Definition

We say $u(x,t), (x,t) \in \mathbb{R} \times [0,\infty)$ is piecewise smooth if

- u is C^1 in all points (x,t) except along a C^1 curve $s(t), t \in (a,\infty)$,
- u is discontinuous in $s(t)$ for all $t \in (a,\infty)$.

In addition we assume that for every $t \in (a,\infty)$ the limits

$$u^+(s(t),t) := \lim_{x \downarrow s(t)} u(x,t) \;\;\&\;\; u^-(s(t),t) := \lim_{x \uparrow s(t)} u(x,t) \quad \text{exist.}$$

Scalar Conversation Law

$$u_t + (f(u))_x = u_t + f'(u)u_x = 0 \text{ in } \mathbb{R} \times [0,\infty) \;\&\; u(x,0) = g(x) \tag{1}$$

where $g \in C^1(\mathbb{R})$ and $f \in C^2(\mathbb{R})$ with $f'' \geq 0$.

Definition (Distributional solutions)

We say a piecewise smooth function u is a solution of (1) in the distributional sense if

$$\int_0^\infty \int_{-\infty}^\infty [u\phi_t + f(u)\phi_x]\, dxdt = 0 \text{ for any } \phi \in C_c^\infty(\mathbb{R} \times (0,\infty)).$$

Theorem (Rankine-Hugoniot jump condition(s))

Let $s(t), t \geq 0$ *is a* C^1 *curve in* $\mathbb{R} \times [0, \infty)$ *(parametrized as graph). Assume u is piecewise smooth in the sense of the previous definition. Then u is a solution of* (1) *in the sense of distributions if and only if u is a classical solution in any point where u is* C^1 *and*

$$s'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} \circ s(t) \quad \textit{for every } t \in (0, \infty).$$

1.6 Non-uniquness and stability of distributional solutions, Lax entropy condition

We consider the Burgers' equation with **discontinuous** initial value:

$$u_t + uu_x = 0 \text{ in } \mathbb{R} \times [0, \infty) \;\;\&\;\; u(x,0) = g(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Let us apply the previous theorem. We want find a C^1 curve $s(t),\ t \geq 0$ that satisfies the jump condition. For the burgers equation we have $f(x) = \frac{1}{2}x^2$. Then the jump condition is

$$\frac{(u^+)^2 - (u^-)^2}{2(u^+ - u^-)} \circ s(t) = \frac{1}{2}(u^+ + u^-) \circ s(t) = s'(t).$$

Therefore, a distributional solutions of the previous PDE with this initial condition is

$$u(x,t) = \begin{cases} 1 & \text{for } x \leq \frac{1}{2}t, \\ 0 & \text{for } x > \frac{1}{2}t. \end{cases}$$

On the other hand, consider

$$u_t + uu_x = 0 \text{ in } \mathbb{R} \times [0, \infty) \;\; \& \;\; u(x,0) = g(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

A distributional solution is

$$v(x,t) = \begin{cases} 0 & \text{for } x \leq \frac{1}{2}t, \\ 1 & \text{for } x > \frac{1}{2}t \end{cases}$$

but also

$$w(x,t) = \begin{cases} 0 & \text{for } x \leq 0, \\ \frac{x}{t} & \text{for } 0 < x < \frac{1}{2}t, \\ 1 & \text{for } x \geq \frac{1}{2}t \end{cases}$$

is a solution that is even continuous. This solution is called the **rarefaction wave**.

For this problem there is no uniqueness.

Question

Which solution should we pick? Which solution is physically meaninful?

For the solution v characteristics emanate from the shock.

This is physically unreasonable.

Recall the characteristcs equations

$$\frac{dx}{dt} = f'(u^r) \;\; \& \;\; \frac{dx}{dt} = f'(u^l).$$

A sufficient condition such that characteristics do not emanate from the shock is

$$f'(u^+) \geq s' \geq f'(u^-). \tag{2}$$

Since f is convex, (2) implies that $u^+ \geq u^-$.

Lax entropy a condition

We say a piecewise smooth solution $u(x, t)$ to a conservation law is an entropy solution if the **Lax entropy condition** (2) holds.

Note that a smooth solution is an entropy solution since there is no curve s that describes a discontinuity.

Theorem

If an entropy solution exists, then it is the unique distributional solution for the scalar conversation law.

2 Linear second order PDEs in 1D

2.1 Classification of linear second order PDEs

Consider linear second order PDE for n independent variables has the form

$$\sum_{i,j=1}^{n} a_{i,j} u_{x_i,x_j} + \sum_{k=1}^{n} b_k u_{x_k} + cu = d \quad \text{on } \Omega. \tag{3}$$

We assume $a_{i,j}, b_k, c, d \in C^0(\Omega)$ and $a_{i,j} = a_{j,i}$. Hence

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \dots & & \dots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \quad \text{is a symmetric matrix.}$$

Recall form linear algebra that there exists a symmetric matrix B such that

$$BAB^\top = \begin{pmatrix} d_1 & 0 & \dots & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots\dots & \dots & \\ 0 & \dots & \dots & d_{n-1} & 0 \\ 0 & \dots & \dots & 0 & d_n \end{pmatrix} =: D$$

with $d_1, \dots, d_n \in C^0(\Omega)$. $\{d_1(\mathbf{x}), \dots, d_n(\mathbf{x})\}$ are the eigenvalues of $A(\mathbf{x})$.

Definition

The PDE (3) is called

1. **Elliptic** if all the eigenvalues $d_1, \dots, d_n$ are positive. That is equivalent to say that A is positive definite,
2. **Parabolic** if exactly one eigenvalue is 0 and the other eigenvalues have the same sign,
3. **Hyperbolic** if exactly one eigenvalue is negative and the other eigenvalues are positive,
4. **Ultrahyperbolic** if there are more thatn one negative eigenvalues and the other eigenvalues are positive.

Consider linear second order PDE for 2 independent variables has the form

$$a_{1,1} u_{x_1,x_1} + 2a_{1,2} u_{x_1,x_2} + a_{2,2} u_{x_2,x_2} + b_1 u_{x_1} + b_2 u_{x_2} + cu = d. \tag{4}$$

The PDE (4) is

1. Elliptic $\iff \quad a_{1,1}a_{2,2} - a_{1,2}^2 > 0$,
2. Parabolic $\iff \quad a_{1,1}a_{2,2} - a_{1,2}^2 = 0$
3. Hyperbolic $\iff \quad a_{1,1}a_{2,2} - a_{1,2}^2 < 0$.

Let B the $n \times n$ matrix such that

$$BAB^\top = D$$

We can introduce new coordinates $(y_1, \ldots, y_n) = \mathbf{y}$ via $B\mathbf{x} = \mathbf{y}$.

Lemma

The PDE (3) writes w.r.t. the coordinates $\mathbf{y}$ *as*

$$\sum_{i=1}^{n} d_i u_{x_i,x_i} + \sum_{k=1} b_k u_{x_k} + cu = d.$$

After rescaling with $\frac{1}{\sqrt{|d_i|}}$ *in* y_i *for every* $i = 1, \ldots, n$ *as long as* $d_i \neq 0$ *this becomes*

$$\Delta u + \nabla u \cdot (b_1, \ldots, b_k) + cu = d.$$

Proof: We compute

$$u_{x_i}(\mathbf{x}) = \frac{\partial u}{\partial x_i}\Big|_{\mathbf{x}} = \frac{\partial(u \circ B^{-1} \circ B)}{\partial x_i}\Big|_{\mathbf{x}} = \nabla_y (u \circ B^{-1})\big|_{B\mathbf{x}} \cdot (B_{1,i}, \ldots, B_{n,i}) = \sum_{k=1}^{n} \frac{\partial u \circ B^{-1}}{\partial y_k}\Big|_{B\mathbf{x}} B_{k,i}$$

We set $u(\mathbf{y}) := u \circ B^{-1}\mathbf{y}$ and $u_{y_i} := \frac{\partial(u \circ B^{-1})}{\partial y_k}$. Therefore

$$u_{x_j,x_i} = \sum_{k,l=1}^{n} u_{y_k,y_l} B_{k,i} B_{l,j} \implies \sum_{i,j=1}^{n} A_{i,j} u_{x_i,x_j} = \sum_{k,l=1}^{n} \underbrace{\sum_{i,j=1}^{n} B_{k,i} A_{i,j} B_{j,l}^\top}_{d_k \delta_{k,l}} u_{y_k,y_l} = \sum_{k=1}^{n} d_k u_{y_k,y_l}.$$

Consider linear second order PDE for 2 independent variables has the form

$$a_{1,1}u_{x_1,x_1} + 2a_{1,2}u_{x_1,x_2} + a_{2,2}u_{x_2,x_2} + b_1 u_{x_1} + b_2 u_{x_2} + cu = d \text{ on } \mathbb{R}. \quad (5)$$

Example

Consider the PDE (5) for 2 independent variables. Let $d = c = b_2 = 0$. Applying the transformations of the previous lemma yields

1. **Elliptic**: $u_{x_1,x_1} + u_{x_2,x_2} + b_1 u_{x_1} = 0$. If $b_1 = 0$, we have the **Laplace equation**:

$$u_{x_1,x_1} + u_{x_2,x_2} = 0.$$

2. **Parabolic**: Assume $d_2 = 0$ and set $x_1 = x$ and $x_2 = t$. Then $u_{x,x} + b_1 u_1$. If $b_1 = 1$ we have the **diffusion equation**:

$$u_{x,x} + u_t = 0.$$

3. **Hyperbolic**: Assume $d_2 < 0$. Then $u_{x,x} - u_{t,t} + b_1 u_t$. If $b_1 = 0$ we have the **wave equation**:

$$u_{x,x} - u_{t,t} = 0.$$