MAT351 Partial Differential Equations
Lecture 8

October 7, 2020

2.2 Wave Equation in 1D

2.2.1 Deriving the wave equation

Consider a flexible, elastic homogenous string or thread of length l, that undergoes relatively small transverse vibrations.

We think of the string as the graph of function $u(x,t)$ on $[0,l]$ that depends also on $t \in [0,\infty)$.

Let $T(x,t)$ be the magnitude of the tension force that pulls in $(x,u(x,t))$ along the string at time t. And let us assume T does not depend on t. Moreover there are no other forces.

Because the string is perfect flexible the tension force is directed tangential to the string.

And let $\rho(x)$ be the mass density of the string as distribution on $[0,l]$. Since the string is homogeneous, we assume $\rho(x) \equiv \mathit{constant}$.

We consider an interval $[x_0,x_1] \subset [0,l]$ that gives a section $\{(x,u(x)) : x \in [x_0,x_1]\}$ of the string.

We apply Newton's law: the Force F is given by mass times accelaration.

This yields the following two equations

$$\frac{T(x_1)}{\sqrt{1+u_x(x_1)^2}} - \frac{T(x_0)}{\sqrt{1+u_x(x_0)^2}} = 0 \quad \text{for horizontal forces}$$

and

$$\frac{T(x_1)u_x(x_1)}{\sqrt{1+u_x(x_1)^2}} - \frac{T(x_0)u_x(x_0)}{\sqrt{1+u_x(x_0)^2}} = \int_{x_0}^{x_1} \rho u_{t,t}(x)dx \quad \text{for vertical forces.}$$

We assume the magnitude of the motion is small compared to 1. By that we mean that the slope $u_x(x,t)$ of $u(x,t)$ w.r.t. x at time t is small compared to 1.

If Taylor expand the $x \mapsto \sqrt{x}$ around 1 we get

$$\sqrt{1+u_x^2} = (1+u_x^2)^{\frac{1}{2}} = \sum_{i=0}^{\infty} \binom{1/2}{i} (u_x)^i = 1 + \frac{1}{2}u_x^2 + \dots.$$

(Binomial series) where

$$\binom{\alpha}{i} = \frac{\alpha \cdot (\alpha-1)\cdot \dots \cdot (\alpha - i + 1)}{1 \cdot 2 \cdot \dots \cdot i} \text{ for } i \in \mathbb{N} \text{ and } \alpha \in \mathbb{R}, \alpha \geq 0.$$

Hence, we make the assumption that Newton's laws for the string reduces to

$$T(x_1) - T(x_0) = 0 \text{ for horizontal forces.}$$

and

$$T(x_1)u_x(x_1) - T(x_0)u_x(x_0) = \int_{x_0}^{x_1} \rho u_{t,t}(x)dx \text{ for vertical forces.}$$

The first equation says that $T(x) \equiv T$ is constant along $[0, l]$.

In the second equation we can apply the fundamental theorem of calculus. Hence

$$T\int_{x_0}^{x_1} u_{x,x}(x,t)dx = \int_{x_0}^{x_1} \rho u_{t,t}(x,t)dx.$$

Since this euqation holds for every $x_0 < x_1$ with x_0 and x_1 close to each other, it follows $Tu_{x,x} = \rho u_{t,t}$. Now, let us also assume the mass ditribution $\rho(x)$ along the string is constant and set $c = \sqrt{\frac{T}{\rho}}$.

Wave equation in 1D

$$u_{t,t} = c^2 u_{x,x} \text{ on } \mathbb{R} \times [0, \infty)$$

for $c \neq 0$.

Modifications

1. If there is an air resistence r present, one has an extra term proportional to the speed u_t:

$$u_{x,x} - c^2 u_{t,t} + r u_t = 0 \quad \text{where} \quad r > 0.$$

2. If there is transversal elastic force, we have an extra term proportional to the magnitude of the displacement u:

$$u_{x,x} - c^2 u_{t,t} + k u = 0 \quad \text{where} \quad k > 0.$$

3. If there is an external force, an extra term f independent of u appears:

$$u_{x,x} - c^2 u_{t,t} + f(x,t) = 0 \quad \text{where} \quad f(x,t) \text{ is a time dependent function.}$$

General solution of the wave equation.

The wave equation in 1D factors nicely in the following way:

$$0 = u_{t,t} - c^2 u_{x,x} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u.$$

We recognize that this yields a coupled system of two first order equations:

$$\begin{aligned} u_t + cu_x &= v \\ v_t - cv_x &= 0. \end{aligned}$$

This idea allows us to prove the following

Theorem

The general C^2 solution of the wave equation $u_{t,t} - c^2 u_{x,x} = 0$ on $\mathbb{R}$ is of the form

$$u(x,t) = f(x+ct) + g(x-ct)$$

for arbitrary functions $f, g \in C^2(\mathbb{R})$.

Proof. We first consider

$$v_t - cv_x = 0 \text{ on } \mathbb{R} \times [0, \infty). \tag{1}$$

We know the general solution of (1):

$$v(x,t) = h(x+ct)$$

for $h \in C^1(\mathbb{R})$ arbitrary where v satisfies $v(x,0) = h(x)$.

Then we can consider

$$u_t + cu_x = h(x+ct) \text{ on } \mathbb{R}\times[0,\infty) \ \text{ with } u(x,0)=\tilde{g}(x),\ g\in C^2(\mathbb{R}). \tag{2}$$

Lets solve this. The characterisitcs equations are

$$\frac{d}{dt}x_{x_0}(t) = c \ \text{ with } x(0)=x_0 \ \ \& \ \ \frac{d}{dt}z_{x_0}(t) = h(x_{x_0}(t)+ct) \ \text{ with } z_{x_0}(0)=\tilde{g}(x_0).$$

It follows that $x_{x_0}(t) = ct + x_0$. Hence for $x\in\mathbb{R}$ and $t>0$ we set $x_0 = x-ct$.
Moreover

$$z_{x_0}(t) = \int_0^t h(x_{x_0}(s)+cs)ds + \tilde{g}(x_0) = \int_0^t h(cs+x_0+cs)ds + \tilde{g}(x_0).$$

Then the solution u in (x,t) is given by $u(x,t) = \displaystyle\int_0^t h(cs+x_0+cs)ds + \tilde{g}(x_0)$.

Applying the substituion ruel $\int_a^b f\circ\phi(s)\phi'(s)ds = \int_{\phi(a)}^{\phi(b)} f(s)ds$ with $\phi(s) = x_0+2cs$ gives

$$u(x,t) = \int_{x_0}^{x_0+2ct}\frac{1}{2c}h(\tau)d\tau + \tilde{g}(x-ct) = \int_{x-ct}^{x+ct}\frac{1}{2c}h(\tau)d\tau + \tilde{g}(x-ct).$$

Then the claim follows with $f(s) := \int_0^s \frac{1}{2c}h(\tau)\tau$ and $g(s) = \int_s^0\frac{1}{2c}h(\tau)d\tau + \tilde{g}(s)$ where $f,g\in C^2(\mathbb{R})$. □

An alternative proof (without characteristics).

We can check that $f(x+ct)$ for

$$f(s) = \int_0^s \frac{1}{2c} h(\tau) d\tau$$

solves the equation $u_t + cu_x = h(x+ct)$. Indeed

$$\frac{\partial}{\partial t} f(x+ct) = f'(x+ct)c = \frac{1}{2} h(x+ct), \quad \frac{\partial}{\partial x} f(x+ct) = f'(x+ct) = \frac{1}{2c} h(x+ct).$$

Therefore

$$\frac{\partial}{\partial t} f(x+ct) + c\frac{\partial}{\partial x} f(x+ct) = h(x+ct).$$

On the other hand $g(x-ct)$ for $g \in C^2(\mathbb{R})$ solves the homogeneous equation $u_t + cu_x = 0$.

But we learned before that the sum of a solution of the homogeneous equation and of a solution of the inhomogeneous equation, still solves the inhomogeneous equation $u_t + cu_x = h(x+ct)$.

Therefore, $f(x+ct) + g(x-ct)$ also solves the wave equation.

Remark

It seems we found two different expression for g (depending on the proof), but for the first expression we fixed a initial condition $\tilde{g}$ and found g depending on $\tilde{g}$.

The Initial Value Problem

Now we consider

$$\begin{cases} u_{t,t} - c^2 u_{x,x} = 0 & \text{on } \mathbb{R} \times [0, \infty) \\ u(x,0) = \phi(x) \ \ \& \ \ u_t(x,0) = \phi(x) & \phi \in C^2(\mathbb{R}), \ \psi \in C^1(\mathbb{R}). \end{cases} \tag{3}$$

Theorem (D'Alembert's formula)

The unique solution of the initial value problem (3) *is given by*

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Proof. From the formula for the general solution, we get

$$\phi(x) = f(x) + g(x) \ \ \& \ \ \frac{1}{c}\psi(x) = f'(x) - g'(x).$$

Differentiating ϕ yields $\phi' = f' + g'$. Adding and substracting these identities yields

$$f'(x) = \frac{1}{2}\left(\phi'(x) + \frac{1}{c}\psi(x)\right), \ \ \& \ \ g'(x) = \frac{1}{2}\left(\phi'(x) - \frac{1}{c}\psi(x)\right).$$

Integrating from

$$f(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int_0^x \psi(s)ds\right) + A_1 \ \& \ g(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int_0^x \psi(s)ds\right) + A_2.$$

Since $\phi(x) = f(x) + g(x)$ we have $A_1 + A_2 = 0$.

Now, we can write

$$\begin{aligned} f(x+ct)+g(x-ct) =&\frac{1}{2}\left(\phi(x+ct)+\frac{1}{c}\int_0^{x+ct}\psi(s)ds\right)+\frac{1}{2}\left(\phi(x-ct)-\frac{1}{c}\int_0^{x-ct}\psi(s)ds\right) \\ &+A_1+A_2 \\ =&\frac{1}{2}\left(\phi(x+ct)+\frac{1}{c}\int_0^{x+ct}\psi(s)ds\right)+\frac{1}{2}\left(\phi(x-ct)+\frac{1}{c}\int_{x-ct}^{0}\psi(s)ds\right) \\ =&\frac{1}{2}\left(\phi(x+ct)+\phi(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds \end{aligned}$$

This is what was to prove. □

- **Semigroup property**: Define

$$W(t)(\phi,\psi):=(u,u_t)$$

where u is the unique solution of the wave equation

$$u_{t,t}=c^2u_{xx} \text{ in } \mathbb{R}\times[0,\infty) \text{ with } \phi(x)=u(x,0),\ \psi(x)=u_t(x,0) \tag{4}$$

given by d'Alembert's formula.

Corollary

The following semi-group property holds

$$W(s+\tau)(\phi,\psi)=W(s)(W(t)(\phi,\psi))$$

Proof. Let $P_1(x, y) = x$ be the projection map.

Then $s \geq 0 \mapsto v(\cdot, s) := P_1 \circ W(s + \tau)(\phi, \psi)$ is a solution of the wave equation with initial conditions

$$v(x, 0) = u(x, \tau) \;\&\; v_t(x, 0) = u_t(x, \tau)$$

By d'Alembert's formula we have $v(x, s) = W(s)(u(\cdot, \tau), u_t(\cdot, \tau)) = W(s)(W(\tau)(\phi, \psi))$. □

- **Causality**: For a point $(x, t) \in \mathbb{R} \times (0, \infty)$ the solution u given by d'Alembert's formula

  $$u(x, t) = \frac{1}{2}\left[\phi(x - ct) + \phi(x + ct)\right] + \frac{1}{2}\int_{x-ct}^{x+ct} \psi(s)d$$

  depends only on the values of ψ on $[x - ct, x + ct]$ and the values of ϕ in $x - ct$ and $x + ct$.

  Moreover, for $s \in (0, t)$ we have by the semi-group property

  $$W(t)(\phi, \psi) = W(t - s)(W(s)(\phi, \psi)).$$

  So $u(x, t)$ also only depends on the values of $W(s)(\phi, \psi) = (u(\cdot, s), u_\tau(\cdot, s))$ on $[x - c(t - s), x + c(t - s)]$.

  Hence, the domain of dependence is space-time triangle in $\mathbb{R} \times [0, \infty)$.

  Similar, the domain of influence for $(x, t) \in \mathbb{R} \times [0, \infty)$ is a space time triangle in $\mathbb{R} \times [0, \infty)$.

Example: Plucked String

Consider the initial value problem

$$u_{t,t} = c^2 u_{x,x} \text{ in } \mathbb{R} \times [0,\infty) \;\;\&\;\; u(x,0) = \phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a. \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right].$$

We note that the initial condition is not C^2-differentiable. Hence also the formula doesn't give us a C^2 solution. Nevertheless, this u is still a solution in a "weak" sense, similar like a distributional solution for conservation laws.

For $t = \frac{a}{2c}$ the solution has the form

$$u(x,t) = \begin{cases} 0 & \text{if } |x| \geq \frac{3a}{2}, \\ \frac{1}{2}\phi(x + \frac{1}{2}a) & \text{if } x \in (-\frac{3a}{2}, -\frac{a}{2}), \\ \frac{1}{2}\left[\phi(x+\frac{1}{2}a) + \phi(x - \frac{1}{2}a)\right] & \text{if } |x| \leq \frac{a}{2} \\ \frac{1}{2}\phi(x - \frac{1}{2}a) & \text{if } x \in (\frac{3a}{2}, \frac{a}{2}). \end{cases}$$

where for instance

$$\frac{1}{2}\phi(x + \frac{1}{2}a) = \frac{1}{2}\left[b - \frac{b}{a}|x + \frac{1}{2}a|\right] = \frac{1}{2}\left[b + \frac{b}{a}x + \frac{1}{2}b\right] = \frac{3}{4}b + \frac{b}{2a}x \text{ for } x \in (-\frac{3a}{2}, -\frac{a}{2}).$$