MAT351 Partial Differential Equations
Lecture 9

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Example: Plucked String

Consider the initial value problem

$$u_{t,t} = c^2 u_{x,x} \text{ in } \mathbb{R} \times [0, \infty) \;\;\&\;\; u(x,0) = \phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a. \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right].$$

For $t = \frac{a}{2c}$ the solution has the form

$$u(x,t) = \begin{cases} 0 & \text{if } |x| \geq \frac{3a}{2}, \\ \frac{1}{2}\phi(x + \frac{1}{2}a) & \text{if } x \in (-\frac{3a}{2}, -\frac{a}{2}), \\ \frac{1}{2}\left[\phi(x + \frac{1}{2}a) + \phi(x - \frac{1}{2}a)\right] & \text{if } |x| \leq \frac{a}{2} \\ \frac{1}{2}\phi(x - \frac{1}{2}a) & \text{if } x \in (\frac{a}{2}, \frac{3a}{2}). \end{cases}$$

where for instance

$$\frac{1}{2}\phi(x + \frac{1}{2}a) = \frac{1}{2}\left[b - \frac{b}{a}|x + \frac{1}{2}a|\right] = \frac{1}{2}\left[b + \frac{b}{a}x + \frac{1}{2}b\right] = \frac{3}{4}b + \frac{b}{2a}x \text{ for } x \in (-\frac{3a}{2}, -\frac{a}{2}).$$

and

$$\frac{1}{2}\phi(x-\frac{1}{2}a)=\frac{1}{2}\left[b-\frac{b}{a}|x-\frac{1}{2}a|\right]=\frac{1}{2}\left[b-\frac{b}{a}x+\frac{1}{2}a\right]=\frac{3}{4}b-\frac{b}{a}x \text{ for } x\in(\frac{a}{2},\frac{3a}{2}).$$

and

$$\begin{aligned}\frac{1}{2}\phi(x+\frac{1}{2}a)+\frac{1}{2}\phi(x-\frac{1}{2}a)&=\frac{1}{2}\left[b-\frac{b}{a}|x+\frac{1}{2}a|\right]+\frac{1}{2}\left[b-\frac{b}{a}|x-\frac{1}{2}a|\right]\\&=\frac{1}{2}\left[b-\frac{b}{a}x-\frac{1}{2}b\right]+\frac{1}{2}\left[b+\frac{b}{a}x-\frac{1}{2}b\right]=\frac{1}{2}b \quad x\in[-\frac{1}{2}a,\frac{1}{2}a].\end{aligned}$$

For $t=\frac{3a}{c}$ the solution has the form

$$u(x,t)=\begin{cases}0 & \text{if } |x|\geq 4a,\\ \frac{1}{2}\phi(x+\frac{1}{2}a) & \text{if } x\in(-4a,-2a),\\ 0 & \text{if } |x|\leq 2a\\ \frac{1}{2}\phi(x-\frac{1}{2}a) & \text{if } x\in(2a,4a).\end{cases}$$

Preservation of Energy

Imagine an infinite string with uniform mass distribution ρ, uniform tension force T given by the graph of $u(x,t)$ for $x \in \mathbb{R}$.
The string behaves according to the wave equation

$$\rho u_{t,t} = T u_{x,x}$$

with initial conditions $\phi(x) = u(x,0)$ and $\psi(x) = u_t(x,0)$ on $\mathbb{R}$. We set again $c = \sqrt{\frac{T}{\rho}}$.

The **kinetic energy** is defined as

$$KE = \frac{1}{2}\int_{-\infty}^{\infty} \rho \cdot (u_t)^2(x)dx.$$

This integral and the following ones are evaluated from $-\infty$ to $+\infty$.

To be sure that the integral converges we assume that $\phi(x) = u(x,0)$ & $\psi(x) = u_t(x,0)$ vanish outside of $[-R,R]$ for some $R > 0$.

Then $u_t(x)$ vanishes outside of $[-R-ct, R+ct]$ and

$$\frac{1}{2}\int_{-\infty}^{\infty} \rho(u_t)^2 dx = \frac{1}{2}\int_{-R-ct}^{R+ct} \rho(u_t)^2 dx.$$

We differentiate the kinetic energy in t:

$$\frac{dKE}{dt} = \frac{1}{2}\rho\int_{-\infty}^{\infty} \frac{d}{dt}(u_t)^2 dx = \rho\int_{-\infty}^{\infty} u_t u_{t,t} dx.$$

Here we apply that we can differentiate under the integral.

At this point we use that u satisfies the wave equation $u_{t,t} = c^2 u_{x,x}$:

$$\frac{dKE}{dt} = T \int_{-\infty}^{\infty} u_t u_{x,x} dx = T u_t u_x \Big|_{-\infty}^{\infty} - T \int_{-\infty}^{\infty} u_{t,x} u_x dx.$$

The first term on the right hand side vanishes since u_t vanishes outside $[-R - ct, R + ct]$.

In the second term we can write $u_{t,x} u_x = \frac{1}{2} \left((u_x)^2\right)_t$. Hence

$$\frac{dKE}{dt} = -\frac{d}{dt} \int_{-\infty}^{\infty} \frac{1}{2} T (u_x)^2 dx.$$

We call $\frac{1}{2} \int_{-\infty}^{\infty} T(u_x)^2 dx =: PE$ the **potential energy**.
We see that

$$\frac{d}{dt} [KE + PE] = 0$$

and therefore

$$KE + PE = \frac{1}{2} \int_{-\infty}^{\infty} \left(\rho (u_t)^2 + T(u_x)^2\right) dx =: E$$

is constant. This is the law of **preservation of enery**.

Application:

We can use the Preservation of Enery to show uniquness of solutions of the wave equation.

Let u^1, u^2 be two solutions of

$$\rho u_{t,t} = T u_{x,x} \quad (\Leftrightarrow \rho u_{t,t} - T u_{x,x} = 0) \tag{1}$$

with the same initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

The wave equation (1) is a linear, homogeneous PDE.

Hence, $v = u^1 - u^2$ is also solution with initial condition $v(x,0) = 0$ and $v_t(x,0) = 0$.

The task is to show that $v(x) \equiv 0$, then clearly we have $u^1 = u^2$.

By Preservation of Energy it follows

$$0 = \frac{1}{2} \int_{-\infty}^{\infty} \left(\rho(v_t(x,0))^2 + T(v_x(x,0))^2\right) dx = \frac{1}{2} \int_{-\infty}^{\infty} \underbrace{\left(\rho(v_t(x,t))^2 + T(v_x(x,t))^2\right)}_{\geq 0} dx.$$

It follows

$$0 = \frac{1}{2} \left(\rho(v_t(x,t))^2 + T(v_x(x,t))^2\right).$$

This is only possible if $v_t \equiv 0$ and $v_x \equiv 0$ ($\Leftrightarrow \nabla v = 0$).

It follows that $v = const$ on $\mathbb{R} \times [0, \infty)$. Since $v(x,0) = 0$, it follows $v(x,t) \equiv 0$.

2.2 The diffusion equation.

Recall the Diffusion Equation

$$u_t = ku_{x,x}. \tag{2}$$

Though similar to the wave equation, its mathematical properties are completely different.

The constant $k > 0$ is called the diffusion constant or **volatility**.

To solve this equation is harder than to solve the wave equation. Therefore we start by assuming we have a solution and studying its properties.

Maximum Principle

Let $u(x, t)$ be a C^2 solution of (2) on $[0, l] \times [0, T] \subset \mathbb{R} \times [0, \infty)$. Then

1. The maximum of $u(x, t)$ is assumed either on $[0, l] \times \{0\}$ or on $\{0\} \times [0, T] \cup \{l\} \times [0, T]$:

$$\max_{(x,t)\in(0,l)\times(0,T]} u(x, t) = \max_{(x,t)\in[0,l]\times\{0\}\cup\{0\}\times[0,T]\cup\{l\}\times[0,T]} u(x, t).$$

   (*Weak Maximum Principle*)

2. If there exists $(x_0, t_0) \in (0, l) \times (0, T]$ such that

$$\max_{(x,t)\in[0,l]\times[0,T]} u(x, t) = u(x_0, t_0) =: M.$$

   Then $u(x, t) \equiv M$ on $[0, l] \times [0, T]$. (*Strong Maximum Principle*)

Maximum Principle, short

Let $M := \max_{(x,t)\in\partial([0,l]\times[0,T])\setminus[0,l]\times\{T\}} u(x,t)$.

1. $u \leq M$ on $(0,l)\times(0,T]$.
2. If there exists $(x_0,t_0)\in(0,l)\times(0,T]$ then $u \equiv M$ on $[0,l]\times[0,T]$.

Mathematical Inerpretation

In this formulation we can see the weak maximum principle as a geometric inequality, and the strong maximum principle as the characterization of the equality case.

Physical Interpretation

Imagine a rod of length $l > 0$ with no internal heat source. Then then the weak maximum principle tells us that the hottest and the coldest spot can only occur at th initial time $t = 0$ or at one of the two ends of the rod.

On the other hand, by the strong maximum principle if the coldest or the hottest spot occur inside of the rod away from the ends at some positive time $t > 0$, then, the temperature distribution must be constant along the rod.

Proof of the weak maximum principle.

Here, we will only prove the weak maximum principle.
The proof of the strong maximum principle is much more difficult and requires tools that currently are not at our disposal.

Idea for the proof of the weak maximum principle:

If there exists $(x_0, t_0) \in (0, l) \times (0, T)$ such that $u(x_0, t_0) = \max_{[0,l]\times[0,T]} u$ then $u_t = u_x = 0$ and $u_{x,x} \leq 0$. If we would even know that $u_{x,x} < 0$, this would contradict the heat equation. For a rigorous proof we need to work a little bit more.

Proof. The trick is to consider $v(x, t) = u(x, t) + \epsilon \frac{1}{2} x^2$ for some $\epsilon > 0$.
Then

$$v_{x,x} = u_{x,x} + \epsilon = k u_t + \epsilon = k v_t + \epsilon.$$

Hence, the Partial Differential Inequality

$$v_{x,x} > k v_t. \tag{3}$$

Let $M = \max_{(x,t)\in\partial([0,l]\times[0,T])\setminus[0,l]\times\{T\}} u(x, t)$. Then it is clear that

$$v(x, t) \leq M + \epsilon l^2 \quad \text{on} \quad \partial([0, l] \times [0, T])\setminus[0, l] \times \{T\}$$

Now suppose that there is $(x_0, t_0) \in (0, l) \times (0, T]$ such that $u(x_0, t_0) > M$. Then

$$v(x_0, t_0) = u(x_0, t_0) + \epsilon \frac{1}{2} x_0^2 > M.$$

In particular, the maximum of v is occurs in a point $(x_1, t_1) \in (0, l) \times (0, T]$ and $v(x_1, t_1) > M$.

If $(x_1, t_1) \in (0, l) \times (0, T)$ then $v_t = 0$ and $v_{x,x} \geq 0$ ($\Rightarrow k v_{x,x} \geq 0$) what contradicts (3).

If $(x_1, t_1) \in (0, l) \times \{T\}$, then we still have $v_{x,x}(x_1, T) \geq 0$.

However $u_t(x_1, T) = 0$ does eventually not hold.

But we now that $v(x_1, T - \delta) \leq v(x_1, T)$. Hence

$$u_t(x_1, T) = \lim_{\delta \downarrow 0} \frac{v(x_1, T - \delta) - v(x_1, T)}{-\delta} \geq 0.$$

So we have

$$v_t(x_1, T) \geq v_{x,x}(x_1, T)$$

what again contradicts (3): $v_{t,t} \leq k u_{x,x}$. $\square$

Application: Uniqueness of solutions to the diffusion equation

Dirichlet Problem for the diffusion equation

We consider the diffusion equation

$$u_t = ku_{x,x} \quad \text{on } [0,l] \times [0,T]$$

with the initial condition

$$u(x,0) = \phi(x) \quad \text{for } x \in [0,l] \quad \text{and } \phi \in C^2([0,l])$$

and boundary values

$$u(0,t) = g(t) \;\&\; u(l,t) = h(t) \quad \text{for } t \in [0,T] \quad \text{and } h, g \in C^2([0,T]).$$

Corollary

There exists at most one C^2 solution to the Dirichlet problem for the diffusion equation.

Proof. The first step of the proof is similar to what we saw for the wave equation.

Assume u^1, u^2 are two solutions to the Dirichlet problem with ϕ, g, h given as above. Then, by linearity of the equation $w = u^1 - u^2$ is a solution of the Dirichlet problem with $\phi = g = h \equiv 0$.

We need to show that $w = 0$.

By the weak maximum principle $w \le 0$. Since also $-w$ solves the Dirichlet problem with $\phi, g, h \equiv 0$ we also have $-w \le 0$. Hence $w = 0$. □

Alternative Proof via enery method

Let us investigate an alternative method to prove uniqueness of solution of the diffusion equation. This method is similar to the strategy that we applied for the corresponding statement for solutions of the wave equation.

Let u be a solution to the previous Dirichlet Problem with $g = h \equiv 0$ and consider

$$E(t) = \int_0^l \frac{1}{2}(u(x,t))^2 dx.$$

Proposition

The qantity $E(t)$ is positive and monotone decreasing in $t \in [0, \infty)$.

Proof. Let us compute the derivative in t.

$$\frac{dE}{dt} = \int_0^l u_t(x,t)u(x,t)dx = \int_0^l u_{x,x}(x,t)u(x,t)dx.$$

By integration by parts the right hand side becomes

$$u_x(l,t)u(l,t) - u_x(0,t)u(0,t) - \int_0^l (u_x(x,t))^2 dx = -\int_0^l (u_x(x,t))^2 dx \leq 0.$$

□

Now, if u^1, u^2 are two solution for the Dirichlet problem with $\phi \in C^2([0,l])$ and $g, h \in C^2([0,T])$, then $u = u^1 - u^1$ is a solution of the Dirichlet problem with $\phi, g, h \equiv 0$.

Since $E(0) = 0$, the previous proposition implies $E(t) = 0$ for all $t \geq 0$ and therefore $u(x,t) \equiv 0$.

Stability

Stability was the third property for well-posedness of a PDE: "Small changes of the date imply only small changes for the corresponding solutions".

The energy method shows the following: If u^1 and u^2 are solutions of the Dirichlet problem with the same g, h and but for ϕ^1 and ϕ^2 that are eventually different, then

$$\int \frac{1}{2}(u^1(x,t) - u^2(x,t))^2 dx \leq \int_0^l \frac{1}{2}(\phi^1(x) - \phi^2(x))^2 dx.$$

This is **Stability in the square integral sense**.

Alternative we can use the maximum principle again. Let u^1 and u^2 be solutions of the Dirichlet problem with ϕ^1, g^1, h^1 and ϕ^2, g^2, h^2 respectively. Set $u = u^1 - u^2$. Then

$$u^1 - u^2 = \leq \max\{\max(\phi^1 - \phi^2), \max(g^1 - g^2), \max(h^1 - h^2)\}.$$

But we also get

$$u^2 - u^1 \leq \max\{\max(\phi^2 - \phi^1), \max(g^2 - g^1), \max(h^2 - h^1)\}.$$

Then

$$|u^1 - u^2| \leq \max\{\max|\phi^2 - \phi^1|, \max|g^2 - g^1|, \max|h^2 - h^1|\}.$$

This is called **stability in the uniform sense.**