MAT351 Partial Differential Equations
Lecture 10

October 19, 2020

Solving the Diffusion Equation on the real line

In this lecture we will solve

The diffusion equation on the real line

$$u_t = ku_{x,x} \text{ on } \mathbb{R} \times (0, \infty) \tag{1}$$

More precisely, we look for $u \in C^2(\mathbb{R} \times (0, \infty))$ that solves the initial value problem

$$\begin{aligned} u_t &= ku_{x,x} && \text{on } \mathbb{R} \times (0, \infty) \\ u(x, 0) &= \phi(x) && \text{for } x \in \mathbb{R}. \end{aligned} \tag{2}$$

$\phi \in C^1(\mathbb{R})$ and $k > 0$.

The initial condition is understood in the sense that $\lim_{t \downarrow 0} u(x, t) = \phi(x)$.

For ϕ we assume that $\phi(x) \to 0$ if $|x| \to \infty$.

The method to find a solution will be very different from the previous techniques that we used.

Let us collect some general properties of solutions of the diffusion equation $u_t = ku_{x,x}$.

(a) **Translation invariance:** If $u(x, t)$ solves (1), then also $u(x - y, t)$ solves (1) for any $y \in \mathbb{R}$.

(b) If $u(x, t)$ is a smooth (C^k) solution (1), any derivative ($u_t, u_x, u_{x,x}$, ect.) if it exists, solves (1) as well.

(c) **Superposition:** Any linear combinations of solutions of (1) is again a solutions of (1):

$$u^i(x,t),\ \ i=1,\ldots,n \text{ solves } (1) \implies \sum_{i=1}^{n} \lambda_i u^i(x,t) =: u(x,t) \text{ solves } (1).$$

(d) **An integral of a solution is again a solution:**If $u(x,t)$ solves (1) and $\phi \in C^0(\mathbb{R})$ then

$$v(x,t) = \int u(x-y,t)\phi(y)dy \ \text{ solves } (1).$$

Proof. We calculate

$$v_t(x,t) = \int \frac{\partial}{\partial t} u(x-y,t)\phi(y)dy = \int \frac{\partial^2}{\partial z^2}\Big|_{z=x-y} u(z,t)\phi(y)dy$$
$$= \int \frac{\partial^2}{\partial x^2}\left[u(x-y)\right]\phi(y)dy = v_{x,x}(x,t). \qquad \square$$

(e) **Scaling property:** If $u(x,t)$ is a solution of (1), so is $u(\sqrt{a}x, at)$ for any $a > 0$.

Remark

Of course these transformations do not preserve the initial value problem

Let us consider the following *special initial condition*:

$$\psi(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

and arbitrary value in $x = 0$.
We consider this ψ because it is scaling invariant: $\psi(ax) = \psi(x)\ \forall a > 0$.

We say $u(x,t)$ in $C^2(\mathbb{R} \times (0,\infty))$ solves the diffusion equation with initial condition ψ if

$$u_t = ku_{x,x} \quad \mathbb{R} \times (0,\infty)$$

and $\lim_{t \downarrow 0} u(x,t) = \psi(x)$ for all $x \in \mathbb{R}$.

If $u(x,t)$ solves the diffusion equation on $\mathbb{R} \times (0,\infty)$ with initial condition $\psi(x)$, then by the scaling property also $u(\sqrt{a}x, at)$ is a solution with the same initial condition $\psi(x)$.

Moreover, we expect uniqueness of solutions for a given initial value function. Hence, it should hold

$$u(x,t) = u(\sqrt{a}x, at).$$

From this we make the following **Ansatz**:

$$Q(x,t) = g(x/\sqrt{t})$$

Why do we choose this Q?

Because Q satisfies $Q(\sqrt{a}t, a) = Q(x,t)$.

Lemma

$Q(x,t) = g(x/\sqrt{t})$ solves (1) if and only if g satisfies $g''(r) = -\frac{1}{2k} r g'(r)$.

Proof. We calculate

$$Q_t(x,t) = \left[g\left(\frac{x}{\sqrt{t}}\right)\right]_t = -\frac{1}{2}\frac{x}{(\sqrt{t})^3} g'\left(\frac{x}{\sqrt{t}}\right)$$

$$Q_x(x,t) = \left[g\left(\frac{x}{\sqrt{t}}\right)\right]_x = \frac{1}{\sqrt{t}} g'\left(\frac{x}{\sqrt{t}}\right)$$

$$Q_{x,x}(x,t) = \left[g\left(\frac{x}{\sqrt{t}}\right)\right]_{x,x} = \frac{1}{t} g''\left(\frac{x}{\sqrt{t}}\right).$$

Hence

$$0 = \frac{1}{t}\left[\frac{1}{2}\frac{x}{\sqrt{t}} g'\left(\frac{x}{\sqrt{t}}\right) + k g''\left(\frac{x}{\sqrt{t}}\right)\right].$$

Since $t > 0$ and by substitution of $\frac{x}{\sqrt{t}}$, it follows that g must satisfy

$$g''(r) = -\frac{1}{2k} r g'(r). \tag{3}$$

On the other hand, if g satisfies (3), then $Q(x,t) = g\left(\frac{x}{\sqrt{t}}\right)$ satisfies the $u_t = k u_{x,x}$. □

Lemma

The general solution of $g''(r) = -\frac{1}{2k} r g'(r)$ *is* $g(r) = c_1 \int_{r_0}^{r} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau + c_2, \;\; c_1, c_2 \in \mathbb{R}.$

Proof. We set $h = g'$ and consider the ODE $h'(r) = -\frac{1}{2k} r h(r)$.

We can easily solve this equation by standard techniques. The general solution is given by

$$h(r) = c_1 e^{-r^2}. \quad \text{And therefore } g(r) = c_1 \int_{r_0}^{r} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau + c_2. \quad \square$$

Corollary

The function $Q(x,t) = c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)} d\tau + c_2$ *is a solution of* $u_t = k u_{x,x}$ *on* $\mathbb{R} \times (0, \infty)$.

We want to choose the constants $c_1, c_2 \in \mathbb{R}$ such that

$$\lim_{t \downarrow 0} Q(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

We can compute the following limits

$$x > 0, \quad \lim_{t \downarrow 0} Q(x,t) = \lim_{t \downarrow 0} c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)} d\tau + c_2$$

$$= \lim_{t \downarrow 0} c_1 \sqrt{4k} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\tau^2} d\tau + c_2 = c_1 \frac{\sqrt{4k\pi}}{2} + c_2.$$

Hence, we require

$$1 = c_1 \frac{\sqrt{4k\pi}}{2} + c_2.$$

Here we used that $\int_0^\infty e^{-\tau^2} d\tau = \frac{\sqrt{\pi}}{2}$. Similar

$$x < 0, \quad \lim_{t \downarrow 0} Q(x,t) = \lim_{t \downarrow 0} c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau + c_2 = -c_1 \frac{\sqrt{4k\pi}}{2} + c_2$$

and hence we require also

$$0 = -c_1 \frac{\sqrt{4k\pi}}{2} + c_2.$$

We can solve this system of two linear equations for c_1 and c_2 and obtain

$$c_1 = \frac{1}{\sqrt{4k\pi}} \text{ and } c_2 = \frac{1}{2}.$$

Corollary

$Q(x,t) = \frac{1}{\sqrt{4k\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau + \frac{1}{2}$ solves $u_t = k u_{x,x}$ on $\mathbb{R} \times (0,\infty)$ and

$$\lim_{t \downarrow 0} u(x,t) = \psi(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases}$$

The fundamental solution

We define

$$S(x,t) = \frac{\partial}{\partial x} Q(x,t) = \frac{\partial}{\partial x} \frac{1}{\sqrt{4k\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau = \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{x}{\sqrt{4kt}}\right)^2}.$$

Not that Q (and S) are C^∞ functions on $\mathbb{R} \times (0,\infty)$ (because e^x is C^∞).

Definition (Fundamental solution)

The function $S(x,t)$ is called the fundamental solution of $u_t = ku_{x,x}$ on the real line.

Theorem

The unique solution of the initial value problem (2) *:*

$$\begin{aligned} u_t &= ku_{x,x} && \text{on } \mathbb{R} \times [0,\infty) \\ u(x,0) &= \phi(x) && \text{for } x \in \mathbb{R} \end{aligned}$$

where $\phi \in C^1(\mathbb{R})$ *with* $\phi(x) \to 0$ *if* $|x| \to \infty$ *and* $k > 0$ *is*

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy.$$

Proof.

We already saw that $u(x,t)$ is indeed a solution of $u_t = ku_{x,x}$ on $\mathbb{R} \times (0,\infty)$.

We only need to check the initial value condition. For that we compute the following:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial z}\Big|_{z=x-y} Q(z,t)\phi(y)dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left[Q(x-y,t)\right]\phi(y)dy$$
$$= -Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=\infty} + \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy.$$

Since $\phi(x) \to 0$ for $|x| \to \infty$, it follows

$$u(x,t) = \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy, \quad t > 0.$$

Moreover

$$\lim_{t\downarrow 0} u(x,t) = \int_{-\infty}^{\infty} \lim_{t\downarrow 0} Q(x-y,t)\phi'(y)dy$$
$$= \int_{-\infty}^{\infty} 1_{[0,\infty)}(x-y)\phi'(y)dy = \int_{-\infty}^{\infty} 1_{(-\infty,0]}(y-x)\phi'(y)dy$$
$$= \int_{-\infty}^{x} \phi'(y)dy = \phi(x).$$

Uniqueness follows by the enery method. □