MAT351 Partial Differential Equations
Lecture 11

October 21, 2020

Last lecture:

Initial value problem for the diffusion equation:

$$\begin{aligned} u_t &= ku_{x,x} \quad \text{on } \mathbb{R} \times (0, \infty) \\ u(x,0) &= \phi(x) \quad \text{for } x \in \mathbb{R}. \end{aligned} \tag{1}$$

$\phi \in C^1(\mathbb{R})$ and $k > 0$. For ϕ we assumed that ϕ is integrable and $\phi(x) \to 0$ if $|x| \to \infty$.

To solve this initial value problem we first solved the following problem

$$\begin{aligned} u_t &= ku_{x,x} \quad \text{on } \mathbb{R} \times (0, \infty) \\ u(x,0) &= \psi(x) \quad \text{for } x \in \mathbb{R} \end{aligned}$$

where

$$\psi(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

and arbitrary value in $x = 0$.

The initial value conditions are understood in the sense that $\lim_{t \downarrow 0} u(x,t) = \phi(x)$ and $\lim_{t \downarrow 0} u(x,t) = \psi(x)$ respectively.

For the second problem we constructed a solution "by hand":

$$Q(x,t) = \frac{1}{\sqrt{4k\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau + \frac{1}{2}$$

Then we defined

Fundamental solution of diffusion equation on $\mathbb{R}$

$$S(x,t) = \frac{\partial}{\partial x} Q(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{x}{\sqrt{4kt}}\right)^2} \quad t > 0 \text{ and } x \in \mathbb{R}.$$

The function $S(x,t)$ is known as *Green's function, the source function or propagator of the diffusion equation on* $\mathbb{R}$.

Theorem

A solution of the initial value problem (**??**) *is given by*

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy.$$

The function

$$S(x-y,t) = p(x,y,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{x-y}{\sqrt{4kt}}\right)^2}$$

is also know as the *heat or diffusion kernel on* $\mathbb{R}$.

Proof.

We already saw that $u(x,t)$ is indeed a solution of $u_t = ku_{x,x}$ on $\mathbb{R} \times (0,\infty)$.

We only need to check the initial value condition. For that we compute the following:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial z}\Big|_{z=x-y} Q(z,t)\phi(y)dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left[Q(x-y,t)\right]\phi(y)dy$$
$$= -Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=\infty} + \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy.$$

Since $\phi(x) \to 0$ for $|x| \to \infty$, it follows

$$u(x,t) = \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy, \quad t > 0.$$

Moreover

$$\lim_{t\downarrow 0} u(x,t) = \int_{-\infty}^{\infty} \lim_{t\downarrow 0} Q(x-y,t)\phi'(y)dy$$
$$= \int_{-\infty}^{\infty} 1_{[0,\infty)}(x-y)\phi'(y)dy = \int_{-\infty}^{\infty} 1_{(-\infty,0]}(y-x)\phi'(y)dy$$
$$= \int_{-\infty}^{x} \phi'(y)dy = \phi(x).$$

□

Properties of the fundamental solution

- $S(x,t) > 0$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$.
- We compute that
$$\int_{-\infty}^{\infty} S(x-y,t)dy = \int_{-\infty}^{\infty} S(y,t)dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4kt}} e^{-\left(\frac{y}{\sqrt{4kt}}\right)^2} dy = \frac{1}{\sqrt{\pi}} \int e^{-x^2} dx = 1.$$
- The solution $u(x,t) = \int S(x-y,t)\phi(y)$ is in $C^\infty(\mathbb{R} \times (0,\infty))$.
- We have
$$\max_{\{x \in \mathbb{R} : |x| \geq \delta\}} S(x,t) \leq \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{\delta}{\sqrt{4kt}}\right)^2} \to 0 \text{ when } t \downarrow 0.$$
  In particular $S(x,t) \to 0$ for all $x \neq 0$ as $t \downarrow 0$.
- $S(x,0)$ is not defined.

  But we computed
$$\lim_{t\downarrow 0} u(x,t) = \lim_{t \downarrow 0} \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \phi(x).$$
  Hence, we can interpret $y \mapsto S(x-y,0)$ not as function but as a linear operators δ_x on $C^1(\mathbb{R})$:
$$\delta_x(\phi) = \phi(x).$$
  The operator δ_x is an example for a **distribution**, the *Dirac δ_x distribution*.

Distributions

Distributions are generalized functions.

We define $\mathcal{D} = C_c^\infty(\mathbb{R})$, the set of C^∞ functions ϕ with $\phi(x) = 0$ for $|x| \geq R$ for some $R > 0$. We say ϕ has compact support.

Definition (Distributions)

A distribution is continuous linear map $\mathcal{F} : \mathcal{D} \to \mathbb{R}$.

What means continuous in this context?

Let us first define a notion of convergence on $\mathcal{D}$. Consider $\phi_i, \phi \in \mathcal{D}$, $i \in \mathbb{N}$.

We say $\phi_i \to \phi$ in $\mathcal{D}$ if

$$\max_{x\in\mathbb{R}} |\phi_i(x) - \phi(x)| \to 0 \text{ and } \max_{x\in\mathbb{R}} \left| \frac{d^k}{dr}\phi_i(x) - \frac{d^k}{dr}\phi(x) \right| \to 0, \quad \forall k \in \mathbb{N}.$$

Then we say a linear map $\mathcal{F} : \mathcal{D} \to \mathbb{R}$ is continuous if

$$\mathcal{F}(\phi_n) \to \mathcal{F}(\phi) \text{ whenever } \phi_n \to \phi \text{ in } \mathcal{D}.$$

The concept of distribution allows us to make sense of $''S(x,0)''$ as the distribution δ_0.

Examples

1. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Then

$$\mathcal{F}(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

is a distribution.

We see $\mathcal{F}$ is linear. Let's check continuity of $\mathcal{F}$. Consider $\phi_n \to \phi \in \mathcal{D}$. Then

$$|F(\phi_n) - \mathcal{F}(\phi)| \leq \int |f(x)| \, |\phi_n(x) - \phi(x)| \, dx \leq \max_{x \in \mathbb{R}} |\phi_n(x) - \phi(x)| \int |f(x)| dx \to 0.$$

Hence $\mathcal{F}$ is indeed a distribution.

The example shows that we can think of functions as distributions.

2. Let f be as before. Then

$$\mathcal{G}(\phi) = \int_{-\infty}^{\infty} f(x)\phi'(x)dx$$

is a distribution. Continuity follows as in the previous example.

Now, if $f \in C^1(\mathbb{R})$, then $\mathcal{G}(\phi) = -\int_{-\infty}^{\infty} f'(x)\phi(x)dx$.

The function f' represents the distribution $\mathcal{G}$.

The concept of distributions now allows us to define derivatives for functions that are not differentiable in the classical sense.

Derivatives in distributional sense

Definition

We say a locally integrable function $f : \mathbb{R} \to \mathbb{R}$ has a derivative in distributional sense if there exists a distribution $\mathcal{G} : \mathcal{D} \to \mathbb{R}$ such that

$$\int f(x)\phi'(x)dx = \mathcal{G}(\phi) \ \forall \phi \in \mathcal{D}.$$

Example

- Every C^1 function has a derivative in distributional sense:

$$\int f(x)\phi'(x)dx = -\int f'(x)\phi(x)dx$$

  and the right hand side defines a distribution.

- The function $f(x) = 0$ for $x < 0$ & $f(x) = x$ for $x \geq 0$ has a derivative in distributional sense:

$$\int f(x)\phi'(x)dx = \int_0^\infty x\phi(x)dx = x\phi(x)\Big|_0^\infty - \int_0^\infty \phi(x)dx = -\int_{-\infty}^\infty \psi(x)\phi(x)dx$$

  and derivative is represented by ψ.

Question: Has the function

$$\psi(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases}$$

a distributional derivative? If yes, what is it?

We can compute the distributional derivative using the $S(x,t)$:

$$\int \psi(x)\phi'(x)dx = \int_0^\infty \phi'(x) = \lim_{t\downarrow 0} \int Q(x,t)\phi'(x)dx = \lim_{t\downarrow 0} \int S(x,t)\phi(x)dx = \phi(0) = \delta_0(\phi)$$

So indeed ψ has a derivative in distributional sense but it cannot be represented as function!

Physical Intepretations of $S(x, t)$.

The fundamental solution $S(x - y, t)$ describes the diffusion of a substance.

For any time $t > 0$ the total mass is 1.

Initally at time $t = 0$ the substance completely concentrated in y.

We can see the convolution $\int S(x - y, t)\phi(y)dy$ also as follows. For $t > 0$ we can approximate the integral via a Riemann sum:

$$\int_{-\infty}^{\infty} S(x - y, t)\phi(y)dy \sim \sum_{i=1}^{n} S(x - y_i, t)\phi(y_i)\Delta y_i$$

where $\{y_0 \leq y_1 \leq \cdots \leq y_n\} \subset \mathbb{R}$ with $n \in \mathbb{N} \uparrow \infty$ and $\Delta y_i = y_i - y_{i-1}$.

On the right hand side we have a sum that is the mean value in space of the family

$$S(x - y_i, t) \text{ weighted with } \phi(y_i), i = 1, \ldots, n.$$

Consequently, we can interpret $\int S(x - y, t)\phi(y)dy$ as the limit of these mean values when we let the number of points go to infinity.

Probabilistic interpretation of $S(x,t)$

.

The fundamental solution is the transition probability density of Brownian motion in $\mathbb{R}$.

What does that mean? If a particle in 0 at time $t = 0$ follows a "random path" then

$$\int_a^b S(-y,t)dy$$

is the probability that we will find this particle at time $t > 0$ in the interval $[a,b]$.

Back to Uniqueness

Lemma

Let $\phi \in C^1(\mathbb{R})$ with $\phi(x) \to 0$ for $|x| \to \infty$. We consider

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy.$$

Then, for $t > 0$ fixed, $u(x,t) \to 0$ for $|x| \to \infty$.

Proof. We pick $\epsilon > 0$. Let $R(\epsilon) > 0$ such that $|\phi(x)| \leq \epsilon$ for $|x| \geq R$ for $R \geq R(\epsilon)$.We fix such an $R > R(\epsilon)$. We pick $R > R(\epsilon)$ such that

$$S(R,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{R}{\sqrt{4kt}}\right)^2} \leq \epsilon.$$

Consider a sequence $(x_n)_{n\in\mathbb{N}}$ with $|x_n| \to \infty$, and let $N \in \mathbb{N}$ such that $|x_n| \geq 2R$ for $n \geq N$.

Let $n \geq \mathbb{N}$. Then

$$u(x_n,t) = \int_{x_n-R}^{x_n+R} S(x_n-y)\phi(y)dy + \int_{\{y\in\mathbb{R}:|x_n-y|>R\}} S(x_n-y)\phi(y)dy.$$

We write $\{y \in \mathbb{R} : |x_n - y| > R\} = \{|x_n - y| > R\}$ in the following.

Hence

$$|u(x_n,t)| \leq \left| \int_{x_n-R}^{x_n+R} S(x_n-y)\phi(y)dy \right| + \left| \int_{\{|x_n-y|>R\}} S(x_n-y)\phi(y)dy \right|.$$

The second integral on the right hand side can be estimated as follows

$$\left| \int_{\{|x_n-y|>R\}} S(x_n-y)\phi(y)dy \right| \leq \int_{\{|x_n-y|>R\}} \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{R}{\sqrt{4kt}}\right)^2} |\phi(y)|\, dy$$
$$\leq \epsilon \int_{\{|x_n-y|>R\}} |\phi(y)|\, dy \leq \epsilon \int |\phi(y)|\, dy.$$

If $|x_n - y| \leq R$, then $|y| \geq |x_n| - |x_n - y| \geq |x_n| - R \geq 2R - R = R$.Therefore, the first integral on the right hand side becomes

$$\left| \int_{x_n-R}^{x_n+R} S(x_n-y)\phi(y)dy \right| \leq \epsilon \int_{x_n-R}^{x_n+R} S(x_n-y)dy \leq \epsilon \int S(x_n-y,t)dy \leq \epsilon.$$

We can conclude that for $n \geq N$ it follows that

$$|u(x_n,t)| \leq \epsilon + \epsilon \int |\phi(y)|dy$$

Therefore

$$\limsup_{n\to\infty} |u(x_n,t)| \leq \epsilon(1 + \int |\phi(y)|dy)\ \forall \epsilon > 0 \Rightarrow \limsup_{n\to\infty} |u(x_n,t)| \leq 0 \ \Rightarrow \lim_{n\to\infty} |u(x_n,t)| = 0.$$

□

Theorem (Existence and Uniqueness)

The initial value problem

$$\begin{array}{rcll} u_t & = & ku_{x,x} & \text{on } \mathbb{R}\times(0,\infty) \\ u(x,0) & = & \phi(x) & \text{for } x\in\mathbb{R} \end{array} \quad (2)$$

for $\phi \in C^1(\mathbb{R})$ with $\phi(x)\to 0$ if $|x|\to\infty$ and $k>0$ has a unique solution $u(x,t)$ with $u(x,t)\to 0$ if $|x|\to\infty$.

Proof. Assume there are 2 solutions $u^1(x,t)$ and $u^2(x,t)$ with $u^1(x,t), u^2(x,t)\to 0$ for $|x|\to\infty$.

Then we consider $u = u^1 - u^2$ and also $u(x,t)\to 0$ if $|x|\to\infty$.

Now we apply the energy method

$$\frac{d}{dt}\int \frac{1}{2}[u(x,t)]^2 dx = \int u_t(x,t)u(x,t)dx = \int ku_{x,x}(x,t)u(x,t)dx.$$

Hence

$$\frac{d}{dt}\int \frac{1}{2}[u(x,t)]^2 dx = ku_x(x,t)u(x,t)\Big|_{x=-\infty}^{x=\infty} - \int (u_x(x,t))^2\,dx \le 0.$$

It follows that

$$\int \frac{1}{2}[u(x,t)]^2 dx \le \int \frac{1}{2}[u(x,\epsilon)]^2 \to 0.$$

Hence $u=0$ and $u^1=u^2$. □