MAT351 Partial Differential Equations
Lecture 12

October 26, 2020

Equations with a source term

In the following we will study inhomogeneous, linear second order PDEs

For instance, consider the initial value problem for diffusion equation with a source term:

Diffusion equation with a source term

Let $f \in C^0(\mathbb{R} \times (0, \infty))$.

$$\begin{aligned} u_t - k u_{x,x} &= f(x,t) \quad \text{on} \quad \mathbb{R} \times (0,\infty) \\ u(x,0) &= \phi(x) \quad \text{on} \quad \mathbb{R}. \end{aligned}$$

The physical interpretation of this equation is, for instance, the heat evolution of an infinitely long rod with an initial temperatur ϕ and a source (or sink) of heat at later times.

Remark

If we define $A = k \frac{\partial^2}{\partial x^2}$, then A is linear operator that goes from $C^2(\mathbb{R})$ to $C^0(\mathbb{R})$.
Then the inhomogeneous diffusion equation then takes the form

$$\frac{d}{dt} u(t) = Au(t) + f(t) \; t > 0 \text{ and } u(0) = \phi \in C^1(\mathbb{R})$$

where $u(t) = u(\cdot, t) \in C^2(\mathbb{R})$ and $f(t) = f(\cdot, t) \in C^0(\mathbb{R})$.

Structural similarities with inhomogeneous ODEs

Recall the following ODE problem.

Let $A \in \mathbb{R}^{n \times n}$.

$$\frac{d}{dt} v(t) = Av(t) + f(t), \quad v(0) = v_0$$

where $t \in [0, \infty) \mapsto v(t), f(t) \in \mathbb{R}^n$.

For $f \equiv 0$ this is a homogeneous, linear ODE with constant coefficients.

The solution is given by $t \in [0, \infty) \mapsto e^{tA} v_0$.

e^{tA} is called the solution operator.

Recall: In case $A = BDB^{-1}$ for a diagonal matrix $D = (d_1, \ldots, d_n)$ then

$$e^{tA} = B(e^{td_1}, \ldots, e^{td_n})B^{-1}$$

More general, one can find the operator e^{tA} by means of the Jordan form for the matrix A.

The solution formula for the inhomogeneous problem with $f \neq 0$ is given by

$$u(t) = e^{tA} v_0 + \int_0^t e^{(t-s)A} f(s) ds.$$

Dunhamel's principle

The solution formula for the inhomogeneous ODE is derived via Dunhamel's principle.

Assume $v(t)$ solves the inhomeogeneous problem. Assume $S(-t) = e^{-tA} = \left[e^{tA}\right]^{-1}$ exists.

Then we can compute

$$S(-t)f(t) = S(-t)\left[\frac{d}{dt}v(t) - Av(t)\right] = S(-t)\frac{d}{dt}v(t) - S(-t)Av(t) = \frac{d}{dt}\left[S(-t)v(t)\right].$$

The last equality is the product rule. Integrating from 0 to $t > 0$ gives

$$\int_0^t S(-s)f(s)ds = S(-t)v(t) - v_0$$

Hence

$$v(t) = S(t)v_0 + S(t)\int_0^t S(-s)f(s)ds = S(t)v_0 + \int_0^t S(t-s)f(s)ds.$$

We can then check that this $v(t)$ indeed satisfies the inhomogeneous ODE

$$\begin{aligned}\frac{d}{dt}v(t) &= \frac{d}{dt}S(t)v_0 + \frac{d}{dt}\int_0^t S(t-s)f(s)ds \\ &= AS(t)v_0 + S(0)f(t) + \int_0^t AS(t-s)f(s)ds \\ &= A\left[S(t)v_0 + \int_0^t S(t-s)f(s)ds\right] + f(t) = Av(t) + f(t).\end{aligned}$$

The solution formula for ODEs gives us an idea how a solution formula for PDEs should look like.

We saw a version of this formula before in the case of inhomogeneous first order PDEs of the form

$$u_t - au_x = f(x,t)$$

Here, the operator is given by $A = a\frac{\partial}{\partial x}$. Then the PDE takes the form

$$u_t = Au + f(x,t)$$

Recall the solution of the homogeneous equation was given by

$$\phi(x+ta) = [S(t)\phi](x)$$

Dunhamel's principle suggests the solution formula

$$v(t) = S(t)\phi + \int_0^t S(t-s)f(s)ds = \phi(x+at) + \int_0^t f(x+a(t-s),s)ds.$$

for the inhomogeneous problem.

This is exactly the formula that we already derived from the method of characteristics.

Back to the inhomogeneous diffusion equation

The unique solution of the initial value problem

$$\begin{aligned} u_t &= k u_{x,x} \quad \text{on } \mathbb{R}\times(0,\infty) \\ u(x,t)\to 0, &\quad |x|\to\infty \\ u(x,0) &= \phi(x) \quad \text{on } \mathbb{R}. \end{aligned}$$

was given by

$$\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy =: \mathbf{S}(t)\phi(x) \;\text{ where } S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\left(\frac{x}{\sqrt{4kt}}\right)^2}.$$

We can see $\mathbf{S}(t) : C^1(\mathbb{R}) \to C^2(\mathbb{R})$ as a family of solution operators.

Now we consider the same problem but with a source term $f \in C^0(\mathbb{R}\times(0,\infty))$:

$$u_t = k u_{x,x} + f(x,t) \quad \text{on } \mathbb{R}\times(0,\infty).$$

We also write $f(s)$ for $f(\cdot, s)$. We assume $|f(x,t)| \leq C$.We prove the following theorem.

Theorem

The unique solution ofthe inhomogeneous problem is given by the formula

$$v(x,t) = [\mathbf{S}(t)\phi](x) + \int_0^t [\mathbf{S}(t-s)f(s)](x)ds.$$

Proof. We only check the existence statement.

First we compute $v_t = [\mathbf{S}(t)\phi]_t + \frac{d}{dt}\int_0^t \mathbf{S}(t-s)f(s)ds = k\,[\mathbf{S}(t)\phi]_{x,x} + \frac{d}{dt}\int_0^t \mathbf{S}(t-s)f(s)ds$.

We consider the second term on the right hand side

$$\frac{d}{dt}\int_0^t [\mathbf{S}(t-s)f(s)](x)ds = \frac{d}{dt}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds = \frac{d}{dt}\int_0^t g(s,t)ds.$$

$s \in (-\infty, t] \mapsto g(s,t)$ is continuous with

$$g(t,t) = \lim_{s\uparrow t} g(s,t) = \delta_x[f(\cdot,t)] = f(x,t).$$

More precisely, we can compute

$$\begin{aligned} g(s,t) &= \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dy \\ &= \int_{-\infty}^{\infty} S(x-y,t-s)f(y,t)dy + \int_{-\infty}^{\infty} S(x-y,t-s)(f(y,s)-f(y,t))dy \end{aligned}$$

For the first term it follows by computation as we did before that

$$\lim_{s\to t}\int_{-\infty}^{\infty} S(x-y,t-s)f(y,t)dy = \delta_x[f(t)] = f(x,t).$$

For the second we get

$$\begin{aligned} \int_{-\infty}^{\infty} S(x-y)[f(y,s)-f(y,t)]dy = &\int_{x-\tilde{C}}^{x+\tilde{C}} S(x-y,t-s)[f(y,s)-f(y,t)]dy \\ &+ \int_{\{y:|x-y|>\tilde{C}\}} S(x-y,t-s)[f(y,s)-f(y,t)]dy \end{aligned}$$

Since $f \in C^0(\mathbb{R} \times (0,\infty))$, f is uniformily continuous on $[x-C, x+C] \times [t-\eta, t+\eta]$ for $t > 0$ and $\eta > 0$ sufficiently small such that $t - \eta > 0$.

In particular, given $\epsilon > 0$ there exists $\delta(\tilde{C}, \epsilon) > 0$ such $|f(y,s) - f(y,t)| < \epsilon$ for $|s-t| < \delta$.

Therefore, for the first term on the right hand side in the last formula we have

$$-\epsilon \leq \int_{x-\tilde{C}}^{x+\tilde{C}} S(x-y, t-s)[f(y,s) - f(y,t)]dy \leq \epsilon.$$

For the second term on the right hand side in the last formula we have

$$-2CKe^{-\tilde{C}^2} \leq \int S(x-y, t-s)[f(y,s) - f(y,t)] \leq 2CKe^{-\tilde{C}^2}$$

because $|f(y,s) - f(y,t)| \leq C$ and $S(x-y, t-s) \leq Ke^{-\tilde{C}^2}$ on $\{y : |x-y| \geq C\}$ for a constant $K > 0$. So we can choose $\tilde{C}$ such that $2CKe^{-\tilde{C}} \leq \epsilon$.

This considerations together imply that

$$\lim_{s \to t} g(s,t) = f(x,t) \pm 2\epsilon$$

and since $\epsilon > 0$ was arbitrary, the limit is $f(x,t)$.

Hence $\tau \in [0, \infty) \mapsto \int_0^\tau g(s,t)ds = \int_0^\tau \int_{-\infty}^{\infty} S(x-y, t-s)f(y,s)dyds$ is differentiable in t with

$$\frac{d}{dt}\int_0^\tau g(s,t)ds = g(t,t) = f(x,t).$$

Therefore

$$\frac{d}{dt}\int_0^t g(s,t)ds = f(x,t) + \int_0^t \frac{\partial}{\partial t} g(s,t)ds$$

For the second term on the right hand side we calculate

$$\int_0^t \frac{\partial}{\partial t} g(s,t)dt = \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y, t-s)f(s)dyds = \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y, t-s)f(s)dyds$$
$$= k\frac{\partial^2}{\partial x^2}\int_0^t \int_{-\infty}^{\infty} S(x-y, t-s)f(s)dyds = k\left[\int_0^t [\mathbf{S}(t-s)f(s)](x)ds\right]_{x,x}.$$

So we computed $v_t = f(x,t) + k\left[\mathbf{S}(t)\phi + \int_0^t \mathbf{S}(t-s)f(s)ds\right]_{x,x}$. □

Diffusion on the half line, Reflection method

We consider the Dirichlet problem for the diffusion equation:

$$\begin{aligned} u_{t,t} &= k u_{x,x} \quad \text{on } \mathbb{R}\times(0,\infty) \\ u(x,0) &= \phi(x) \quad \text{on } [0,\infty) \\ v(0,t) &= 0 \quad \text{for } t>0. \end{aligned} \tag{1}$$

To find a solution formula for this equation we apply the reflection method.
Consider the odd extension of ϕ to the real line:

$$\phi_{odd}(x) = \begin{cases} \phi(x) & x \geq 0, \\ -\phi(-x) & x < 0. \end{cases}$$

The corresponding initial value problem has the solution

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{odd}(y)dy.$$

Since ϕ_{odd} is odd, also $u(x,t)$ is odd, that is $u(x,t) = -u(-x,t)$ (Exercise). Hence $u(0,t) = 0$ and the restriction v of u to $[0,\infty)\times[0,\infty)$ satisfies the Dirichlet boundary condition.
Hence v solves the Dirichlet problem for the diffusion equation with initial condition ϕ.
A formula of v that only depends on ϕ (and not on ϕ_{odd} or $S(x,t)$ for $x\in\mathbb{R}$) is derived as follows

$$v(x,t) = \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_{-\infty}^{0} S(x-y,t)\phi_{odd}(y)dy$$

$$= \int_0^{\infty} \left[S(x-y,t)\phi(y) + S(x-y,t)\phi(-y)\right]dy = \int_0^{\infty} \left[S(x-y,t) - S(x+y,t)\right]\phi(y)dy.$$

The solution of the problem (1) is given by the formula

$$v(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)]\, \phi(y) dy.$$

Similar, we can consider the Neumann problem for the diffusion equation:

$$\begin{aligned} u_t &= k u_{x,x} \quad \text{on } [0,\infty) \times (0,\infty) \\ u(x,0) &= \phi(x) \quad \text{on } \mathbb{R} \\ v_x(0,t) &= 0 \quad \text{for } t > 0. \end{aligned}$$

To derive a solution formula we apply the same strategy as for the Dirichlet problem:
We consider the following initial value problem for the diffusion equation on the real line:

$$\begin{aligned} u_{t,t} &= k u_{x,x} \quad \text{on } \mathbb{R} \times (0,\infty) \\ u(x,0) &= \phi_{even}(x) \quad \text{on } \mathbb{R} \end{aligned}$$

where

$$\phi_{even}(x) = \begin{cases} \phi(x) & x \geq 0 \\ \phi(-x) & x < 0 \end{cases}$$

The solution of this initial value problem will be again even in x: $u(x,t) = u(-x,t)$.

Diffusion with source term on the half line

Now we consider

$$\begin{aligned} u_t - ku_{x,x} &= f(x,t) \quad \text{on } [0,\infty)\times(0,\infty) \\ u(x,0) &= \phi(x) \quad \text{on } \mathbb{R} \\ u(0,t) &= h(t) \quad \text{for } t>0. \end{aligned} \tag{2}$$

for a boundary source function $h : [0,\infty) \to \mathbb{R}$ in $C^1([0,\infty))$.

A strategy to solve this problem is the *Substraction method*:

We consider $v(x,t) = u(x,t) - h(t)$. If $u \in C^2((0,\infty)\times(0,\infty))$ solves the previous problem, then $v \in C^2((0,\infty)\times(0,\infty))$ solves

$$\begin{aligned} v_t - kv_{x,x} &= f(x,t) - h'(t) \quad \text{on } [0,\infty)\times(0,\infty) \\ v(x,0) &= \phi(x) - h(0) \quad \text{on } \mathbb{R} \\ u(0,t) &= 0 \quad \text{for } t>0. \end{aligned}$$

To solve this problem we can apply the reflection method as we did for the equation with $f \equiv 0$.

Then one can check that $v(x,t) + h(t) =: u(x,t)$ solves the problem (2).