MAT351 Partial Differential Equations
Lecture 13

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Diffusion on the half line, Reflection method

We consider the Dirichlet problem for the diffusion equation:

$$\begin{aligned} u_t &= k u_{x,x} \quad \text{on } (0,\infty)\times(0,\infty) \\ u(x,0) &= \phi(x) \qquad \text{on } [0,\infty) \\ u(0,t) &= 0 \qquad \text{for } t>0. \end{aligned} \tag{1}$$

To find a solution formula for this equation we apply the **reflection method**:

Consider the odd extension of ϕ to the real line:

$$\phi_{odd}(x) = \begin{cases} \phi(x) & x \geq 0 \\ -\phi(-x) & x < 0. \end{cases}$$

The corresponding initial value problem has the solution:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{odd}(y)dy.$$

Since ϕ_{odd} is odd, also $x \mapsto u(x,t)$ is odd, that is $u(x,t) = -u(-x,t)$ (Exercise).

Hence $u(0,t) = 0$ and the restriction v of u to $[0,\infty)\times[0,\infty)$ solves the Dirichlet problem for the diffusion equation with initial condition ϕ.

A solution formula of v that only depends on ϕ is derived as follows

$$v(x,t) = \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_{-\infty}^{0} S(x-y,t)\phi_{odd}(y)dy$$

$$= \int_0^{\infty} \left[S(x-y,t)\phi(y) + S(x+y,t)\phi_{odd}(-y)\right] dy = \int_0^{\infty} \left[S(x-y,t) - S(x+y,t)\right]\phi(y)dy.$$

The solution of the problem (1) is given by the formula

$$v(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)]\, \phi(y) dy.$$

Similar, we can consider the Neumann problem for the diffusion equation:

$$\begin{aligned} u_t &= k u_{x,x} && \text{on } [0,\infty)\times(0,\infty) \\ u(x,0) &= \phi(x) && \text{on } [0,\infty) \\ u_x(0,t) &= 0 && \text{for } t>0. \end{aligned}$$

To derive a solution formula we apply the same strategy as for the Dirichlet problem.
We consider the following initial value problem for the diffusion equation on the real line:

$$\begin{aligned} u_t &= k u_{x,x} && \text{on } \mathbb{R}\times(0,\infty) \\ u(x,0) &= \phi_{even}(x) && \text{on } \mathbb{R} \end{aligned}$$

where ϕ_{even} is the even extension of ϕ to $\mathbb{R}$:

$$\phi_{even}(x) = \begin{cases} \phi(x) & x \geq 0 \\ \phi(-x) & x < 0 \end{cases}$$

The solution of this initial value problem will be again even in x: $u(x,t) = u(-x,t)$.

Diffusion with source term on the half line

Now we consider

$$\begin{aligned} u_t - ku_{x,x} &= f(x,t) \quad \text{on } (0,\infty)\times(0,\infty) \\ u(x,0) &= \phi(x) \quad \text{on } [0,\infty) \\ u(0,t) &= h(t) \quad \text{for } t>0. \end{aligned} \tag{2}$$

for a boundary source function $h : [0,\infty) \to \mathbb{R}$ in $C^1([0,\infty))$.

A strategy to solve this problem is the *Substraction method*:

We consider $v(x,t) = u(x,t) - h(t)$. If $u \in C^2((0,\infty)\times(0,\infty))$ solves the previous problem, then $v \in C^2((0,\infty)\times(0,\infty))$ solves

$$\begin{aligned} v_t - kv_{x,x} &= f(x,t) - h'(t) \quad \text{on } [0,\infty)\times(0,\infty) \\ v(x,0) &= \phi(x) - h(0) \quad \text{on } [0,\infty) \\ u(0,t) &= 0 \quad \text{for } t>0. \end{aligned}$$

To solve this problem we can apply the reflection method as we did for the equation with $f \equiv 0$.

Then one can check that $v(x,t) + h(t) =: u(x,t)$ solves the problem (2).

Wave equation with a source term

Consider $\phi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$ and $f \in C^0(\mathbb{R} \times (0,\infty))$ and the inital value problem

$$\begin{aligned} u_{t,t} - c^2 u_{x,x} &= f(x,t) \quad \text{on } \mathbb{R} \times (0,\infty), \\ u(x,0) &= \phi(x) \quad \text{on } \mathbb{R}, \\ u_t(x,0) &= \psi(x) \quad \text{on } \mathbb{R}. \end{aligned} \tag{3}$$

We can interpret f as an external force that acts on an infinitely long vibrating string.

We will prove

Theorem

The unique solution of the initial value problem (3) *is*

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\iint\limits_{\Delta_{x,t}} f(y,s)dyds.$$

The double integral in the formula is on the characteristic space-time triangle $\Delta_{x,t}$ corresponding to the point $(x,t) \in \mathbb{R} \times (0,\infty)$. More precisely

$$\frac{1}{2c}\iint_{\Delta_{x,t}} f(y,s)dyds = \frac{1}{2c}\int_0^t \int_{x-ct}^{x+ct} f(y,s)dyds.$$

Deriving the solution formual via the operator method

We follow the same ideas as for the diffusion equation.

Defining the operator $A = c\frac{\partial}{\partial x}$ the PDE takes the form

$$\begin{aligned} \frac{d}{dt}u - A^2 u &= f(t) \quad \text{on } \mathbb{R}\times(0,\infty), \\ u(0) &= \phi \quad \text{on } \mathbb{R}, \\ \frac{d}{dt}u(0) &= \psi \quad \text{on } \mathbb{R}. \end{aligned}$$

where $u(t) = u(\cdot,t) \in C^2(\mathbb{R})$ and $f(t) = f(\cdot,t) \in C^0(\mathbb{R})$ for $t > 0$.

This equation has again the structure of an ODE of the form

$$\begin{aligned} \frac{d^2}{dt^2}u - a^2 u &= f(t) \quad \text{on } (0,\infty), \\ u(0) &= \phi \in \mathbb{R} \\ u'(0) &= \psi \in \mathbb{R}. \end{aligned} \tag{4}$$

where $f \in C^0([0,\infty))$. Let us first consider the case $f \equiv 0$.

We consider the solutions u_1 and u_2 for problem with the following initial conditions

$$\begin{aligned} u_1(0) &= 0 \\ u_1'(0) &= \psi \end{aligned} \quad \text{and} \quad \begin{aligned} u_2(0) &= \phi \\ u_2'(0) &= 0 \end{aligned}$$

and the sum $u_1 + u_2 = u$ is a solution of (4).

Precisely $u_1(t) = \psi\frac{1}{a}\sin(at)$, $u_2(t) = \phi\cos(at)$and $u = \psi\frac{1}{a}\sin(at) + \phi\cos(at)$.

We can define the solution operator

$$S(t)\psi = \psi \frac{1}{a} \sin(at) = u_1(t) \quad \text{and} \quad \frac{d}{dt}[S(t)\phi] = \phi \cos(at) = u_2(t).$$

We note that $S(0)\psi = 0$ and $\frac{d}{dt}\Big|_{t=0}[S(t)\phi] = \phi$.

By Dunhammel's principle the general solution for the inhomogeneous ODE

$$\begin{aligned} \frac{d^2}{dt^2} u - a^2 u &= f(t) \neq 0 \qquad \text{on } \mathbb{R} \times (0, \infty), \\ u(0) &= 0 \in \mathbb{R} \\ u'(0) &= \psi \in \mathbb{R}. \end{aligned} \tag{5}$$

is given by the formula

$$\tilde{u}_1(t) = S(t)\psi + \int_0^t S(t-s) f(s) ds.$$

Indeed, since we can check that $\frac{d^2}{dt^2}[S(t)\psi] - a^2 S(t)\psi = 0$ and

$$\frac{d^2}{dt^2} \int_0^t S(t-s) f(s) ds = f(t) - a^2 \left[\int_0^t S(t-s) f(s) ds \right]$$

Now, by linearity

$$\tilde{u}_1 + u_2 = S(t)\psi + \int_0^t S(t-s) f(s) ds + \frac{d}{dt} S(t)\phi = v$$

solves the inhomogeneous problem with $v(0) = \phi$ and $v'(0) = \psi$.

The same method works for the wave equation with source. First we solve the IVP (3) with $f \equiv 0$. By d'Alembert's formula the solution is

$$u_1(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy = [\mathbf{S}(t)\psi](x) \text{ for } \phi = 0 \text{ and } \psi \in C^1(\mathbb{R})$$

where $[\mathbf{S}(0)\psi](x) = 0$ and

$$u_2(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] \text{ for } \phi \in C^2(\mathbb{R}) \text{ and } \psi = 0.$$

Observe that $u_2(x,t) = \frac{d}{dt}\mathbf{S}(t)\phi(x)$ with $\frac{d}{dt}\big|_{t=0}[\mathbf{S}(t)\phi](x) = \phi(x)$. Then

$$u_1 + u_2 = \frac{d}{dt}\mathbf{S}(t)\phi + \mathbf{S}(t)\psi = u$$

solves the initial value problem for the homogeneous wave equation and

$$v(x,t) = \frac{d}{dt}\mathbf{S}(t)\phi + \mathbf{S}(t)\psi + \int_0^t \mathbf{S}(t-s)f(s)(x)ds$$

is the "candidate" for a solution of the initial value problem of the inhomogeneous wave equation.

This is the formula that shows up in the theorem before. Indeed

$$\int_0^t \mathbf{S}(t-s)f(s)(x)ds = \int_0^t \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t+s)} f(y,s)dyds.$$

Proof of the theorem

By linearity of the PDE we only need to check that the function

$$(x,t) \in \mathbb{R} \times (0,\infty) \mapsto \frac{1}{2c} \iint_{\Delta_{x,t}} f(y,s)dyds = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

satisfies the inhomogeneous wave equation with initial conditions $\phi = \psi = 0$.

We apply the coordinate change

$$\xi = x + ct, \quad \eta = x - ct.$$

First, we note that the operator $\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$ becomes $4c^2 \frac{\partial^2}{\partial \xi \partial \eta}$. Indeed let $\tilde{g}(\xi, \eta)$ be defined by $\tilde{g}(\xi, \eta) = \tilde{g}(x + ct, x - ct) = g(x,t)$. We compute

$$\frac{\partial}{\partial x} g(x,t) = \frac{\partial}{\partial \xi} \tilde{g}(\xi,\eta) + \frac{\partial}{\partial \eta} g(\xi,\eta), \qquad \frac{\partial}{\partial t} g(x,t) = c \frac{\partial}{\partial \xi} \tilde{g}(\xi,\eta) - c \frac{\partial}{\partial \eta} g(\xi,\eta).$$

It is straightforward to confirm that

$$g_{t,t} - c^2 g_{x,x} = -4c^2 \frac{\partial^2}{\partial \xi \partial \eta} \tilde{g}(\xi,\eta).$$

Using the transformation formula we compute the integral

$$\frac{1}{2c} \iint_{\Delta_{x,t}} f(y,s)dyds = \frac{1}{2c} \iint_{\Delta_{x.t}} \tilde{f}(y + cs, y - cs)dyds.$$

The Jacobian determinant of the transformation $\Phi(x,t) = (x+ct, x-ct)$ is

$$|\det D\Phi(x,t)| = \left|\det\begin{pmatrix}1 & c\\ 1 & -c\end{pmatrix}\right| = 2c.$$

Hence

$$\frac{1}{2c}\iint_{\Delta_{x,t}} f(y,s)dyds = \frac{1}{4c^2}\iint_{\Delta_{x,t}} \tilde{f}\circ\Phi(y,s)J\Phi(y,s)dyds = \frac{1}{4c^2}\iint_{\Phi(\Delta_{x,t})}\tilde{f}(\xi,\eta)d\xi d\eta.$$

where

$$\frac{1}{4c^2}\iint_{\Phi(\Delta_{x,t})}\tilde{f}(\xi,\eta)d\eta d\xi = \frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi}\tilde{f}(\xi,\eta)d\eta d\xi = -\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\xi}^{\eta_0}\tilde{f}(\xi,\eta)d\xi d\eta.$$

Hence

$$-4c^2\frac{\partial^2}{\partial\eta_0\partial\xi_0}\frac{-1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\xi}^{\eta_0}\tilde{f}(\xi,\eta)d\xi d\eta = \frac{\partial}{\partial\eta}\int_{\xi_0}^{\eta_0} f(\xi_0,\eta)d\eta = \tilde{f}(\xi_0,\eta_0) = f(x,t).$$

Hence, we confirmed the PDE. □

Consequences: Wellposedness of the wave equation with a source term

Existence follows from the solution formula.

Uniqueness Let u be a C^2 solution on $\mathbb{R} \times [0, \infty)$ for the wave equation with source and initial values $\phi = \psi = 0$. Then

$$\frac{1}{2c} \iint_{\Delta_{x,t}} f(y,s) dyds = \frac{1}{2c} \iint_{\Delta_{x,t}} [u_{t,t} - c^2 u_{x,x}] dyds.$$

By the divergence theorem it follows

$$= \frac{1}{2c} \int_{\partial \Delta_{x,t}} N \cdot (-c^2 u_x, u_t) dL = \frac{1}{2c} \int_{\partial \Delta_{x,t}} N \cdot (-c^2 u_x, u_t) dL.$$

This line integral has 3 components: the bottome side

$$\frac{1}{2} \int_{x-ct}^{x+tc} -c u_t(y, 0) dy = 0$$

and the side formed by curve $s \in [0,t] \mapsto x + c(t-s)$. Note that the normal vector on this side is $\frac{1}{\sqrt{c^2+1}}(1, c)$ and line integral along this curve comes with a weight $\sqrt{c^2+1}$.

$$\frac{1}{2c} \int_0^t c u_t(x + c(t-s) - c^2 u_x(x + c(t-s), s) ds$$
$$= \frac{1}{2} \int_0^t \frac{d}{ds}[u(x + c(t-s), s)] ds = \frac{1}{2}[u(x,t) - u(x+ct, 0)] = \frac{1}{2} u(x,t).$$

Similar for the remaining term. Hence

$$u(x,t) = \frac{1}{2c} \iint_{\Delta_{x,t}} f(y,s) dyds.$$

Stability: We claim the wave equation with source is stable. That means small perturbations of the data functions f, ϕ and ψ result in small perturbations of the solution u.

How do we measure smallness?

Definition (Maximums Norm on $\mathbb{R}$ and $\mathbb{R} \times [0, \infty)$)

Let $v \in C^0(\mathbb{R})$ and $w \in C^0(\mathbb{R} \times [0, \infty))$. Define the Maximumnorms:

$$\|v\| = \max_{x \in \mathbb{R}} |v(x)|, \qquad \|w\|_T = \max_{(x,t) \in \mathbb{R} \times [0,T]} |w(x,t)|$$

From the solution formula we have the following a priori estimate for the solution u on $\mathbb{R} \times [0, T]$:

$$\begin{aligned} |u(x,t)| &\leq \frac{1}{2}|\phi(x+ct) - \phi(x-ct)| + \frac{1}{2c}\int_{x-ct}^{x+ct} |\phi(y)|dy + \frac{1}{2c}\iint_{\Delta_{x,t}} |f(y,s)|dyds \\ &\leq \|\phi\| + T\|\psi\| + T^2\|f\|_T \end{aligned}$$

Hence

$$\|u\|_T \leq \|\phi\| + T\|\psi\|_T + T^2\|f\|_T .$$

If we have to solution u_1 and u_2 with corresponding data $\phi_1, \phi_2, \psi_1, \psi_2, f_1, f_2$, then $u_1 - u_2$ is a solution with data $\phi_1 - \phi_2, \psi_1 - \psi_2, f_1 - f_2$ by linearity of the problem.

Hence the estimate for the norm yields stabiltiy w.r.t. the Maximums Norm.