MAT351 Partial Differential Equations
Lecture 14

November 2, 2020

Reflection method for wave equations

We will study the following Dirichlet problem for the wave equation on the half-line:

$$\begin{array}{rclcc} v_{t,t} & = & c^2 v_{x,x} & \text{on} & (0,\infty)\times\mathbb{R} \\ v(x,0) & = & \phi(x) & \text{on} & (0,\infty) \\ v_t(x,0) & = & \psi(x) & \text{on} & (0,\infty) \\ v(0,t) & = & 0 & \text{on} & \mathbb{R}. \end{array} \tag{1}$$

The reflexion method works the same way as for the diffusion equation.

We consider *odd* extensions ϕ_{odd} and ψ_{odd} of ϕ and ψ respectively.

Let $u(x,t)$ be the solution of the initial value problem for the wave equation on $\mathbb{R}$. We have the formula

$$u(x,t) = \frac{1}{2}\left[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)dy.$$

Then $u(x,t)$ is once again odd. In particular we have $u(0,t) = 0$ for $t > 0$ and we can define the solution v on $[0,\infty)\times\mathbb{R}$ of (1) by restriction of u to $[0,\infty)$.

We observe that for $x \geq c|t|$ it follows that $x - ct, x + ct \geq 0$. Hence

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy \quad x \geq c|t|.$$

For $0 < x < c|t|$ we have $\phi_{odd}(x - ct) = -\phi(-x + ct)$. Hence

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(-x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{0} [-\psi(-y)]dy + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)dy \quad 0 < x < c|t|.$$

We can apply a change of variable $y \mapsto -y$ to the first integral term. We obtain

$$\begin{aligned} v(x,t) &= \frac{1}{2}\left[\phi(ct+x) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{0} \psi(y)dy + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)dy \\ &= \frac{1}{2}\left[\phi(ct+x) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)dy \quad 0 < x < c|t|. \end{aligned}$$

The complete solution is given by

$$v(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy & \text{if } x \geq c|t| \\ \frac{1}{2}\left[\phi(ct+x) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)dy & \text{if } 0 < x < c|t|. \end{cases}$$

Finite Interval

Similarly we can also study the problem

$$\begin{array}{ccccc} v_{t,t} & = & c^2 v_{x,x} & \text{on} & (0,l)\times\mathbb{R} \\ v(x,0) & = & \phi(x) & \text{on} & (0,l) \\ v_t(x,0) & = & \psi(x) & \text{on} & (0,l) \\ v(0,t)=v(l,t) & = & 0 & \text{on} & \mathbb{R}. \end{array} \tag{2}$$

Diffusion equation with continuous initial data

Let us consider once more

$$\begin{array}{rcll} u_t & = & k u_{x,x} & \text{on } \mathbb{R}\times(0,\infty) \\ \lim_{t\downarrow 0} u(x,t) & = & \phi(x) & \text{on } \mathbb{R} \end{array}$$

This time we assume $\phi \in C^0(\mathbb{R})$ and $|\phi(x)| \leq M\ \forall x \in \mathbb{R}$.

The convolution formula

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-y}{\sqrt{4kt}}\right)^2} \phi(y) dy$$

still makes sense. Indeed, since $|\phi(x)| \leq M$ the integral is finite and bounded from above by M:

$$|u(x,t)| = \left| \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\left(\frac{z}{\sqrt{4kt}}\right)^2} \phi(x-z) dz \right| \leq \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\left(\frac{z}{\sqrt{4kt}}\right)^2} M dz \leq M.$$

A refined satement is that for $m \leq \phi(x) \leq M$ it follows

$$m \leq \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \leq M \quad \forall t > 0 \qquad \textit{(Maximum Principle).}$$

Theorem

Let $\phi(x)$ and $u(x,t)$ be as above. Then $u \in C^\infty(\mathbb{R}\times(0,\infty))$ such that $u_t = k u_{x,x}$ on $\mathbb{R}\times(0,\infty)$ and $\lim_{t\downarrow 0} u(x,t) = \phi(x)$ for every $x \in \mathbb{R}$.

Proof of the theorem

We check that u is in $C^\infty(\mathbb{R}\times(0,\infty))$. Let $S(x,t)=\frac{1}{\sqrt{4\pi kt}}e^{-\left(\frac{x}{\sqrt{4\pi kt}}\right)^2}$.
We show that

$$\frac{\partial}{\partial x}\int S(x-y,t)\phi(y)dy=\int\frac{\partial}{\partial x}S(x-y,t)\phi(y)dy.$$

Recall that

$$\frac{\partial}{\partial x}S(x-y,t)\phi(y)=\lim_{h\to 0}\frac{1}{h}\left[S(x+h-y,t)-S(x-y,t)\right]\phi(y).$$

By the *dominated convergence theorem* for integrals we can pull this limit inside the integral if the modulus of the limit is bounded by an integrable function.This is indeed the case

$$\left|\frac{\partial}{\partial x}S(x-y,t)\phi(y)\right|\leq\left|-\frac{1}{\sqrt{4\pi kt}}\frac{x-y}{2kt}e^{-\frac{(x-y)^2}{4kt}}\right|M\leq\frac{M}{\sqrt{4\pi kt}}\frac{|x-y|}{2kt}e^{-\frac{|x-y|^2}{4kt}}.$$

The term on the right hand side has a finite integral on $\mathbb{R}$. Hence

$$\frac{\partial}{\partial x}u(x,t)=\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}S(x-y,t)\phi(y)dy.$$

All other derivatives of higher order in x and t will work the same way: we always get an estimate by function of the form

$$C|y-x|^n e^{-\tilde{C}(x-y)^2}$$

that has finite integral on $\mathbb{R}$.

Checking the initial condition

We also know that u satisfies $u_t = ku_{x,x}$ because $S(x,t)$ does.

Hence, we only need to prove that u satisfies the initial condition for $t \downarrow 0$.

Consider

$$u(x,t) - \phi(x) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy - \int_{-\infty}^{\infty} S(x-y,t)\phi(x)dy = \int_{-\infty}^{\infty} S(x-y,t)(\phi(y)-\phi(x))$$

Since ϕ is continous in x, for $\epsilon > 0$ we can choose $\delta > 0$ such that

$$|y-x| \leq \delta \;\Rightarrow |\phi(x)-\phi(y)| \leq \epsilon$$

Hence

$$\begin{aligned} |u(x,t)-\phi(x)| &\leq \int_{\{y\in\mathbb{R}:|x-y|>\delta\}} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{2kt}} \underbrace{|\phi(x)-\phi(y)|}_{\leq 2M} dy \\ &\quad + \int_{\{y\in\mathbb{R}:|x-y|\leq\delta\}} S(x-y,t) \underbrace{|\phi(x)-\phi(y)|}_{\leq \epsilon} dy \\ &\leq \frac{2M}{\sqrt{4\pi}} \int_{\{z\in\mathbb{R}:|z|\geq \frac{\delta}{\sqrt{kt}}\}} e^{-\frac{z^2}{4}} dz + \epsilon. \end{aligned}$$

It follows that

$$\limsup_{t\downarrow 0} |u(x,t)-\phi(x)| \leq \epsilon$$

Since $\epsilon > 0$ was arbitrary, it follows that $\lim_{t\downarrow 0} |u(x,t)-\phi(x)| = 0$. □

Additional Remarks

- Decay of the solution for $t \to \infty$.
  For $\phi \in C^0(\mathbb{R})$ with $|\phi| \le M$ we have

$$|u(x,t)| \le \int_{-\infty}^{\infty} S(x-y,t)|\phi(y)|dy \le \frac{M}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} dy \le \frac{M}{\sqrt{4\pi kt}} \to 0.$$

  In particular, this means the backwards diffusion equation

$$u_t = -ku_{x,x} \text{ on } \mathbb{R} \times (0,\infty)$$

  is not well-posed because stability fails.

- About uniqueness again: Let $\phi_1, \phi_2 \in C^0(\mathbb{R})$ with $|\phi_1|, |\phi_2| \le M$.
  We saw that in the class of solutions with $u(x,t) \to 0$ for $|x| \to \infty$ we find a unique solution.
  But if we drop this assumption uniquness might fail: There are solutions of the heat equation with $u(x,t) \to 0$ for $t \downarrow 0$ for all $x \in \mathbb{R}$.
  See also exercise 10 on page 399 in Choksi's Lecture Notes for an example that hints to nonuniquness.