MAT351 Partial Differential Equations
Lecture 15

November 18, 2020

Separation of Variables: Wave equation

Today we begin to study wave and diffusion equations on a finite interval.

First we consider the wave equation on an intervall $[0, l]$ of finite length:

$$\begin{array}{llll} u_{t,t} & = c^2 u_{x,x} & \text{on} & \mathbb{R} \times (0, l) \\ u(0, x) & = \phi(x) & u_t(0, x) = \psi(x) & \text{on } [0, \infty). \end{array} \tag{1}$$

We assume **Dirichlet boundary conditions**

$$\text{DC: } u(t, 0) = u(t, l) = 0 \text{ on } \mathbb{R}.$$

Recall that the PDE is linear and homogeneous. Therefore, if u_1 and u_2 are solutions to (1), then also $u_1 + u_2 = u$ is a solution to (1).
This is called *superpositon principle*.

We will build the general solution for (1) from special ones that are easier to find.

The easier solutions we want to find have the following structure:

$$u(x, t) = X(x) \cdot T(t)$$

(*Separation of variables*).

Assuming this particular structure the PDE reduces to

$$X(x)T''(t) = c^2 X''(x)T(t)$$

This yields

$$-\frac{T''}{c^2 T} = -\frac{X''}{X} = \lambda.$$

for a constant $\lambda \in \mathbb{R}$.

The last equation yields two separate differential equations for T and X:

$$-\frac{T''}{c^2 T} = \lambda \quad \text{and} \quad -\frac{X''}{X} = \lambda.$$

For the moment let us assume $\lambda > 0$.
Why can we do that?

If $\lambda = 0$, we have that $X'' = 0$. It follows that $X(x) = C + Dx$.

By the boundary condition $X(0) = X(l) = 0$ it follows $X \equiv 0$ and $u \equiv 0$.

If $\lambda > 0$ we set $\beta = \sqrt{\lambda} > 0$:

$$T'' + \beta^2 c^2 T = 0 \quad \& \quad X'' + \beta^2 X = 0.$$

We can easily see that the last two equations have the following general solution

$$T(t) = A\cos(\beta ct) + B\sin(\beta ct) \quad \& \quad X(x) = C\cos(\beta x) + D\sin(\beta x).$$

for real constants $A, B, C, D \in \mathbb{R}$.
In particular, any $u = T \cdot X$ with such T and X solves $u_{t,t} = c^2 u_{x,x}$.

Now, we would like to choose the constants A, B, C, D accordingly to given initial and boundary conditions.

For a given time t_0 a solution $u(t_0, x) = T(t_0)X(x)$ must satify the boundary condition:

$$0 = X(0) = C \qquad 0 = X(l) = D\sin(\beta l)$$

We are not interested in the trivial solution with $D = C = 0$.

Hence $\beta l = n\pi$ for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, the roots of the sine function. Or equivalently

$$\lambda_n = (\beta_n)^2 = \left(\frac{n\pi}{l}\right)^2.$$

Hence

$$X_n(x) = \sin\left(\frac{n\pi}{l}x\right), \qquad n \in \mathbb{N},$$

is a family of distinct solutions where $D = 1$.

Note that each sine function may be multiplied with a function that is contant in x to obtain another solution.

We obtain an infinite number of solutions of the form

$$u_n(x,t) = \left(A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right)\sin\left(\frac{n\pi}{l}x\right)$$

for constants $A_n, B_n \in \mathbb{R}$.
Moreover, any **finite** sum of these solutions is also a solution:

$$u(x,t) = \sum_{i=1}^{k}\left(A_{n_i} \cos\left(\frac{n_i\pi}{l}ct\right) + B_{n_i} \sin\left(\frac{n_i\pi}{l}ct\right)\right)\sin\left(\frac{n_i\pi}{l}x\right).$$

Now, assume $\lambda < 0$. We will rule out this case. We set $\beta = \sqrt{-\lambda}$.

Again we can easily see that the general solutions for $T'' + \lambda T = 0$ and $X'' + \lambda X = 0$ are given by

$$T(t) = A\cosh(\beta ct) + B\sinh(\beta ct) \;\;\&\;\; X(x) = C\cosh(\beta x) + D\sinh(\beta x)$$

The boundary condition again implies $0 = X(l) = D\sinh(\beta x)$.

This can only occur if $D = 0$.

A similar argument also rules out the case $\lambda \in \mathbb{C}\backslash(0,\infty) \times \{0\}$ (complex numbers).

Hence, the relevant numbers λ in the problem are positive.

We note that we also could assume **Neumann boundary conditions**

$$\text{NC:} \;\; u_x(t,0) = u_x(t,l) = 0 \;\; \text{on } \mathbb{R}.$$

for the PDE in the beginning.

Then the considerations for λ are similar as for Dirichlet case. We can rule out that $\lambda < 0$.

In the case $\lambda = 0$, the equation for X becomes $X'' = 0$. Again we have

$$X(x) = C + Dx, \quad C, D \in \mathbb{R}$$

Together with the Neumann boundary condition $X_x(0) = X_x(l) = 0$ we see that for any $C \in \mathbb{R}$ the constant function $X(x) = C$ is a solution.

For $\lambda = \beta^2 > 0$ we have the solutions

$$X(x) = C\cos(\beta x) + D\sin(\beta X)$$

The Neumann boundary condition imply that

$$0 = X_x(0) = -C\beta\sin(\beta 0) + D\beta\cos(\beta 0) = D$$

Hence $D = 0$ and $X_x(l) = -C\beta\sin(\beta l)$. Hence, we have again $\beta l = n\pi$ and we define a family of solutions

$$\tilde{X}_n(x) = \cos\left(\frac{n\pi}{l}x\right)$$

where we set $C = 1$.

A family of solutions for the PDE with Neumann boundary conditions is then

$$u_n(x,t) = \left(A\cos(\frac{n\pi}{l}ct) + B\sin(\frac{n\pi}{l}ct)\right)\cos\left(\frac{n\pi}{l}x\right)$$

And again finite sums of these solutions are also solultions

$$u(x,t) = \sum_{i=1}^{k}\left(A\cos(\frac{n_i\pi}{l}ct) + B\sin(\frac{n_i\pi}{l}ct)\right)\cos\left(\frac{n_i\pi}{l}x\right).$$

Finally, we want to bring the inital conditions ϕ and ψ into play.

For this we go back to the Dirichlet condition.

The solution given by the previous formula solves the initial value problem if

$$\phi(x) = u(x,0) = \sum_{i=1}^{k} A_{n_i} \sin\left(\frac{n_i \pi}{l} x\right)$$

and

$$\psi(x) = u_t(x,0) = \sum_{i=1}^{k} \frac{n_i \pi c}{l} B_{n_i} \sin\left(\frac{n_i \pi}{l} x\right)$$

Question

Can we approximate any continuous function ϕ with $\phi(0) = \phi(l)$ by *trigonometric polynomials* of the form

$$\tilde{\phi}(x) = \sum_{i=1}^{k} A_{n_i} \sin\left(\frac{n_i \pi}{l} x\right)$$

What does approximation mean in this context?
And do the solutions w.r.t. $\tilde{\phi}$ approximate the solution w.r.t. ϕ?
Or can we maybe write any continuous function ϕ as series of the form

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l} x\right).$$